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Remarks on the analytic structure of supersymmetric effective actions

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Abstract: We study the effective superpotential of $\mathcal{N} = 1$ supersymmetric gauge theories with a mass gap, whose analytic properties are encoded in an algebraic curve. We propose that the degree of the curve equals the number of semiclassical branches of the gauge theory. This is true for supersymmetric QCD with one adjoint and polynomial superpotential, where the two sheets of its hyperelliptic curve correspond to the gauge theory pseudoconfining and higgs branches. We verify this proposal in the new case of supersymmetric QCD with two adjoints and superpotential $V(X) + XY^2$, which is the confining phase deformation of the D_{n+2} SCFT. This theory has three kinds of classical vacua and its curve is cubic. Each of the three sheets of the curve corresponds to one of the three semiclassical branches of the gauge theory. We show that one can continuously interpolate between these branches by varying the couplings along the moduli space.

Keywords: Supersymmetry and Duality, Confinement, Supersymmetric Effective Theories, Nonperturbative Effects.

Contents

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1. Introduction and summary

$\mathcal{N}=1$ supersymmetric gauge theories are a natural testing ground for the study of nonperturbative gauge dynamics. During the last decade, the tools of holomorphy and symmetries, mainly developed by Seiberg [1], made it possible to gain a deep insight in the strong coupling regime of gauge dynamics. In theories with a mass gap, there is an alternative method, proposed by Dijkgraaf and Vafa [2], that allows to compute systematically the offshell effective superpotential just above the mass gap, where the elementary degree of freedom is the glueball. The vacuum structure of the gauge theory is encoded in an algebraic curve, which in the original formulation is obtained by the planar limit of a related matrix model.

Cachazo, Douglas, Seiberg and Witten [3], by studying the ring of gauge invariant chiral operators, showed that the matrix model loop equations are reproduced in the gauge theory by a set of anomalous Ward identities, which are a generalization of the Konishi anomaly [4]. They considered in particular an $U(N_c)$ supersymmetric gauge theory with a chiral superfield X in the adjoint representation of the gauge group, with tree level superpotential $W = \operatorname{Tr} V(X)$, where $V'(x)$ is a degree n polynomial. The chiral ring of the quantum theory is described by the hyperelliptic Riemann surface

$$y^2 = V'(x)^2 + \hbar f(x)\,, \tag{1.1}$$

where $f(x)$ is a degree $n-1$ polynomial with vanishing classical limit. This surface is a double-cover of the x plane, which describes the expectation values of the adjoint $\langle X\rangle$. The first sheet is visible classically, while the second one is not accessible semiclassically. In the quantum theory, the two sheets are connected by n branch cuts and, at first, the meaning of the "invisible sheet" was not clear. Only when coupling the theory to the chiral superfields in the fundamental representation it was possible to understand the nature of the second sheet [5]. Let us see briefly why, considering $U(N_c)$ adjoint SQCD with tree level superpotential

$$W = \operatorname{Tr} V(X) + \widetilde{Q} m(X) Q\,, \tag{1.2}$$

where $m(x)$ has degree $n-1$ and we suppressed flavor indices. We have two different classical vacua of this theory. In the pseudoconfining vacuum the fundamentals vanish and the adjoint has diagonal expectation values equal to the roots of the adjoint polynomial $V'(x)$. In the higgs vacuum, also Q and $\widetilde{Q}$ acquire an expectation value and the adjoint is equal to the roots of $m(x)$. The gauge group is generically broken to $\prod_{i=1}^{k} U(N_i)$ with $\sum_i N_i = N_c - L$ and $k \leq n$, where L is the number of higgsed colors, and at low energy the nonabelian factors confine, leaving a $\mathrm{U}(1)^k$ theory. In theories with fundamentals, once we fix the number of unbroken gauge groups k, there is no order parameter to distinguish the

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pseudoconfining and higgs phases in an invariant way [6]. Thus one expects that in the full quantum theory the different classical vacua with the same number of unbroken U(1) gauge groups can be connected to each other. So we would use the word *branch* rather than phase to label the pseudoconfining and higgs vacua.

The concept of branches only makes sense in the semiclassical limit of large expectation values. Following [5], we can study the chiral ring, whose generators are the observables

$$M(x) = \widetilde{Q} \frac{1}{x - X} Q \,, \qquad T(x) = \operatorname{Tr} \frac{1}{x - X} \,. \tag{1.3}$$

These observables are meromorphic functions on the Riemann surface (1.1), whose only singularities are simple poles. The classical limit of these operators characterizes the different branches. In the pseudoconfining branch, $M(x)$ and $T(x)$ are regular on the first semiclassical sheet, while on the higgs branch these generators have poles on the first sheet at the higgs eigenvalues of the adjoint $\langle X \rangle$. We can continuously interpolate between the two branches by moving the poles between the two sheets through the branch cuts. Therefore, in this case the first sheet corresponds to the pseudoconfining branch and the second sheet to the higgs branch and the connection between classical phases, or branches, and degree of the curve is clear.

However, more general supersymmetric gauge theories have algebraic curves of higher degree, which give rise to branched coverings of the plane with a larger number of sheets. It is not clear what the meaning of the "invisible sheets" is in general.

In this paper, we suggest that this correspondence between the degree of the curve and the number of branches is a generic feature of $\mathcal{N} = 1$ theories. Consider a supersymmetric gauge theory with a matter content such that, once we fix the number of unbroken U(1)s, there is no order parameter to distinguish between the various classical branches in an invariant way. This is the case of a theory with fundamentals, for instance. Under these assumption, we propose that

An $\mathcal{N} = 1$ supersymmetric gauge theory with a mass gap is described by a degree k algebraic curve, where k is the number of different semiclassical branches of the theory. The curve is a k-sheeted covering of the plane, where each sheet corresponds to a different branch.

Note that we exclude the case in which the theory has a Coulomb branch, as it happens in the theory (1.2) for $n = N$. In this case in fact there is no mass gap.

1.1 SQCD with two adjoints

As a check of our proposal, in this paper we generalize the analysis of [5] to the case of an SU(N_c) SQCD with two adjoint chiral superfields X and Y and a confining phase superpotential

$$W = \operatorname{Tr} V(X) + \lambda \operatorname{Tr} XY^2 + Q\, m(X) Q \,. \tag{1.4}$$

We will find that this theory has three kinds of vacua, the pseudoconfining, the usual "abelian" higgs and a new branch that we will denote "nonabelian higgs phase". The pseudoconfining vacua are the irreps of the equations of motion with vanishing fundamentals. In the one adjoint case we discussed above, we have just one dimensional vacua. In this case, a part from the usual one dimensional vacua $X = a_i \mathbb{1}$ and $Y = b_i \mathbb{1}$, that we will call *abelian* vacua, we have also two dimensional irreps, that we will call *nonabelian* vacua, in which the adjoints are proportional to the Pauli matrices $X = \widehat{a}_i \sigma_3$ and $Y = d_i \sigma_3 + c_i \sigma_1$.[1] The higgs vacua are the ones in which also the fundamentals acquire an expectation value. First of all, there are the usual one dimensional higgs vacua, where X and Y are proportional to the identity, as in the usual abelian vacua. For this reason, we will denote this vacuum the *abelian higgs* branch. But there is also a new kind of higgs vacuum, the *nonabelian higgs* branch, in which the adjoints are two dimensional $X = x_h \sigma_3$ and $Y = y_h \sigma_3 + y_1 \sigma_1$ and the fundamentals are nonvanishing.

Due to the presence of fundamentals, we expect no phase transition and in the full quantum theory the three branches will be connected by continuously varying the couplings. We will study then the chiral ring in the quantum theory, by means of the DV method. In order to compute the curve of the gauge theory, we will use the matrix model loop equations discussed by Ferrari [8], that in the gauge theory are reproduced by a set of generalized Konishi anomaly equations. Our analysis confirms that the DV method works for theories with two adjoint chiral superfields as well as for one adjoint theories. The gauge theory curve, however, is not hyperelliptic as in the usual case (1.1), but *cubic*

$$y^3 + a(x^2) y^2 + b(x^2) y + c(x^2) = 0 \,, \tag{1.5}$$

where the coefficients are even polynomials depending on the couplings and the quantum deformations. This curve is the same as the curve of Ferrari's two matrix model in the planar limit [8].

To have a clear picture of the phase structure of the quantum theory, we will consider again the chiral operators $M(x)$ and $T(x)$ defined in (1.3). One can solve for these operators by the method of anomaly equations and find that they are meromorphic functions on the cubic curve (1.5), whose only singularities are simple poles. In particular, the poles of $T(x)$ have integer residue as in the one adjoint case [5]. We will show that, by moving poles between the three sheets, it is possible to connect continuously all the three branches. Moreover, a natural correspondence arises between the branches and the sheets: we can characterize each of the three branches by specifying the sheet on which $M(z)$ is regular, or by some combination of poles and residues of $T(x)$. In this way, we verify that our proposed correspondence between degree of the curve and the number of branches is satisfied in a very nontrivial way.

1.2 Outline of the paper

The paper is organized as follows. In section 2 we describe the classical chiral ring of SQCD with two adjoints and superpotential (1.4). There are pseudoconfining vacua of two kinds, one as well as two dimensional. Correspondingly, we also have two kinds of higgs vacua, the usual one dimensional and the new two dimensional “nonabelian” higgs vacuum.

In section 3 we approach the quantum theory by means of the DV method and study the cubic equation satisfied by the resolvent $R(x)$. We describe in details the analytic

[1]This phenomenon was first noted in [7] and then discussed in [8] in the case of a supersymmetric gauge theory with adjoint fields and no fundamentals.

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structure of this gauge theory curve as a three sheeted covering of the plane and identify the glueballs of the low energy SQCDs as some particular A periods of the resolvent. In section 4 we solve for the meson operators and get a first idea of the correspondence between branches and number of sheets.

In section 5 we solve for $T(x)$ and argue that its only singularities are simple poles with integer residue. We characterize the three branches by the location of the poles and show that we can continuously interpolate between the different classical vacua of the theory with the same unbroken U(1) factors, generalizing the result of [5] to SQCD with two adjoints. In section 6 then we discuss our proposal that the degree of the gauge theory curve is equal to the number of branches of the semiclassical theory and check it to hold in the case of SQCD with different extra matter. We also comment on the truncation of the chiral ring for n even.

The SQCD with two adjoints has an equivalent Seiberg dual description proposed by Brodie in the superconformal case [9]. In section 7 we generalize this duality to the theory with the confining phase superpotential (1.4), in which case we have one as well as two dimensional pseudoconfining vacua. We will find the classical duality map, in the spirit of KSS, and, in a simple case, also the map in the quantum theory, following [10].

Finally, in section 8 we suggest some other examples in which to test our proposed correspondence between branches and sheets. We present also some speculations about two adjoint SQCD with E_n type superpotentials and its geometric realization.

There are a bunch of appendices. In the first appendix we discuss the generalized Konishi anomalies we used in the main text to solve for the chiral ring. In appendix A.4 we review the solution of cubic algebraic equations. Finally, in appendix C we find a basis for the holomorphic differentials on the cubic curve (1.5) of the gauge theory.

2. The classical theory

In this section we study the classical vacua of the theory. We have the usual pseudoconfining vacua, with vanishing fundamentals, and we distinguish them in abelian ones, that is one dimensional irreps of the algebra of the equations of motion, and nonabelian ones, denoting two dimensional irreps. Then we have the abelian higgs vacua and a new classical phase that we will call nonabelian higgs vacuum. The theory is different depending on whether n is odd or even. In the following we will consider in details the former case. In the latter, the pseudoconfining vacua are still one and two dimensional only, but the chiral ring is not truncated. The analysis of the quantum theory goes through for both cases with analogous treatments.

Consider an $\mathcal{N}=1$ supersymmetric $\mathrm{SU}(N_c)$ gauge theory with matter content consisting in two chiral superfields X and Y in the adjoint representation, N_f fundamentals Q^f and N_f anti-fundamentals $\widetilde{Q}_{\widetilde{f}}$ (f and $\widetilde{f}$ are the flavor indices). We let this theory flow to its IR fixed point and then we turn on the following superpotential

$$W = \mathrm{Tr}\, V(X) + \lambda\, \mathrm{Tr}\, XY^2 + \alpha\, \mathrm{Tr}\, Y + \widetilde{Q}\, m(X) Q\,. \tag{2.1}$$

where we suppressed flavor indices and we introduced the adjoint polynomial

$$V(x)=\sum_{k=1}^{n}\frac{t_k}{k+1}\,x^{k+1}+\beta\,x\,, \tag{2.2}$$

and the meson deformation $m(x)=m_1+m_2\,x$ is diagonal in the flavor indices, while α and β are two Lagrange multipliers enforcing the tracelessness condition.[2] For ease of notations we included the multiplier β as a linear term in the adjoint polynomial (2.2). It will be convenient in the following to separate the odd and even part of the derivative of this polynomial as $V'(x)=-v_+(x^2)-xv_-(x^2)$. As a result, we have cast the multiplier β into the definition of $v_+(x^2)$. The equations of motion are

$$V'(X)+\lambda Y^2+m_2\widetilde{Q}Q=0\,,\qquad \lambda\{X,Y\}+\alpha=0\,, \tag{2.3}$$

$$\widetilde{Q}m(X)=0\,,\qquad m(X)Q=0\,. \tag{2.4}$$

The (2.3) are the X and Y equations of motion, while (2.4) are the equations for the fundamentals. In addition, by varying (2.1) with respect to the Lagrange multipliers we get the tracelessness condition $\mathrm{Tr}\;X=\mathrm{Tr}\;Y=0$.

2.1 Pseudoconfining vacua

We consider at first the pseudoconfining vacua, in which the fundamentals vanish. We want to study the irreducible representations of the algebra defined by the adjoint equations of motion (2.3) for $\langle Q\rangle=\langle\tilde{Q}\rangle=0$. The Casimirs are $X^2=x^2\mathbb{1}$, $Y^2=y^2\mathbb{1}$. Then the first equation reads $\lambda y^2=v_+(x^2)+Xv_-(x^2)$ and we can outline two different cases.

i) abelian vacua The one-dimensional representations are the solutions to

$$\begin{cases} y=-\dfrac{\alpha}{2\lambda x}\,,\\ \lambda y^2+V'(x)=0\,. \end{cases} \tag{2.5}$$

Thus we have $n+2$ vacua

$$\langle X\rangle=\begin{pmatrix} a_1 & & & & \\ & \ddots & & & \\ & & a_2 & & \\ & & & \ddots & \\ & & & & \ddots \end{pmatrix},\qquad \langle Y\rangle=\begin{pmatrix} b_1 & & & & \\ & \ddots & & & \\ & & b_2 & & \\ & & & \ddots & \\ & & & & \ddots \end{pmatrix} \tag{2.6}$$

where the X expectation values a_i are the roots of the degree $n+2$ abelian polynomial

$$p(x)=x^2\,V'(x)+\frac{\alpha^2}{4\lambda}=0\,, \tag{2.7}$$

and $b_i=-\frac{\alpha}{2\lambda a_i}$. Each a_i,b_i has multiplicity N_i such that $\sum_{i=1}^{n+2}N_i=N_c$. By imposing the tracelessness condition on the abelian vacua (2.6) we fix the multipliers α and β. The symmetry breaking pattern is $\mathrm{SU}(N_c)\to\mathrm{U}(1)^{n+1}\times\prod_{i=1}^{n+2}\mathrm{SU}(N_i)$.

[2] The superpotential (2.2) would be irrelevant in the UV for $n>2$, however there always exists a range of flavors N_f such that it is a relevant deformation of the IR fixed point [11, 12].

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ii) nonabelian vacua The only higher dimensional irreps are two dimensional ones, that we parameterize in terms of the Pauli matrices $X = \widehat{a}_i\sigma_3$ and $Y = c_i\sigma_1 + d_i\sigma_3$. To satisfy the X equation of motion, the odd part of the adjoint polynomial must vanish, so we have $\frac{n-1}{2}$ nonabelian vacua $\widehat{a}_i$ which satisfy $v_-(\widehat{a}_i^2) = 0$. The Y expectation values are $d_i = -\frac{\alpha}{2\lambda}\frac{1}{\widehat{a}_i}$ and $c_i = [\lambda^{-1}v_+(\widehat{a}_i^2) - d_i^2]^{\frac{1}{2}}$. The nonabelian vacua display a $\mathbb{Z}_2$ symmetry that acts by reflection of the eigenvalues around the origin. Note also that $x = 0$ is not a solution. Consider the gauge symmetry breaking in the nonabelian vacua, for simplicity consider unbroken gauge group $\mathrm{SU}(N_c)$ with N_c even. The generic nonabelian vacuum is given by

$$\langle X\rangle = \begin{pmatrix} \widehat{a}_1\sigma_3 & & & \\ & \cdot & & \\ & & \widehat{a}_2\sigma_3 & \\ & & & \cdot \\ & & & & \cdot \end{pmatrix}, \qquad \langle Y\rangle = \begin{pmatrix} c_1\sigma_1 + d_1\sigma_3 & & & \\ & \cdot & & \\ & & c_2\sigma_1 + d_2\sigma_3 & \\ & & & \cdot \\ & & & & \cdot \end{pmatrix}, \tag{2.8}$$

where each $\widehat{a}_i$ has multiplicity $\widehat{N}_i$ such that $2\sum_{i=1}^{\frac{n-1}{2}}\widehat{N}_i = N_c$. In this case, unlike the usual one-dimensional one, the vacuum decreases the rank of the gauge group. The gauge symmetry is broken as $\mathrm{SU}(N_c) \to \mathrm{U}(1)^{\frac{n-1}{2}} \times \prod_{i=1}^{\frac{n-1}{2}}\mathrm{SU}(\widehat{N}_i)$.

One can easily show that there are no higher dimensional irreps of the equations of motion (2.3), following [7]. One can shift $X \to X + aY$ and $Y \to Y + bX$ and get to a new algebra with $X^2 = Y^2 = 0$ and $\{X, Y\} + c = 0$. This algebra has just one irreducible representation, which is two dimensional and corresponds to the Fock space of a single fermionic creation-annihilation algebra.[3] The generic gauge symmetry breaking pattern, in the pseudoconfining case, is the following

$$\mathrm{SU}(N_c) \longrightarrow \mathrm{U}(1)^{\frac{3}{2}(n+1)-1} \times \prod_{i=1}^{n+2}\mathrm{SU}(N_i) \times \prod_{i=1}^{\frac{n-1}{2}}\mathrm{SU}(\widehat{N}_i)\,, \tag{2.9}$$

where $N_c = \sum_{i=1}^{n+2}N_i + 2\sum_{i=1}^{\frac{n-1}{2}}\widehat{N}_i$. At energies below the vevs but above the dynamical scale of the theory, we flow to a bunch of $\frac{3}{2}(n+1)$ low energy SQCDs with massive fundamentals, whose number is N_f in the $n+2$ abelian vacua and $2N_f$ in the $(n-1)/2$ nonabelian vacua. At low energies we are left with a $\mathrm{U}(1)^{\frac{3}{2}(n+1)-1}$ theory.

2.2 Higgs vacua

The equations of motion (2.3) allow also for higgs solutions, in which the fundamentals acquire a vacuum expectation value. The Yukawa coupling contains just terms in the dressed X-mesons. There are two different kinds of higgs solutions. The first one is the usual one dimensional vacuum, that we will denote abelian higgs, but it turns out that there are also new two dimensional solutions, analogous to (2.8), that we will denote nonabelian higgs. We consider for simplicity the higgsing of just the last flavor.

[3] This argument holds irrespectively of n, so we have a finite number of vacua both if n is odd and even.

i) abelian higgs The usual one-dimensional higgs vacua are given by

$$\begin{aligned} \widetilde{Q}_{N_f} &= (\widetilde{h}, 0, \ldots, 0)\,, & Q^{N_f} &= (h, 0\ldots, 0)\,, \\ X &= \mathrm{diag}\,(x_h, \text{pseudoconf.})\,, & Y &= \mathrm{diag}\,(y_h, \text{pseudoconf.})\,, \end{aligned} \tag{2.10}$$

where x_h is a root of the meson deformation $m(x)$, i.e. $x_h = -m_1/m_2$, and $y_h = -\frac{\alpha}{2\lambda x_h}$ and the squark expectation values are fixed by the X equations of motion to $\widetilde{h}h = -\frac{1}{m_2}[V'(x_h) + \lambda y_h^2]$. The remaining diagonal expectation values for the adjoints in (2.10) are the generic pseudoconfining vacua (2.6) and (2.8) and we have to impose the tracelessness. The symmetry breaking pattern is as in (2.9) but now the sum of the ranks of the low energy SQCDs decreases by one, $\sum_{i=1}^{n+2} N_i + 2\sum_{i=1}^{\frac{n-1}{2}} \widehat{N}_i = N_c - 1$. In this abelian higgs case we higgs one color direction.

ii) nonabelian higgs The equations of motion (2.3) admit also two-dimensional representations with nonvanishing fundamentals

$$\begin{aligned} \widetilde{Q}_{N_f} &= \left(\widehat{\widetilde{h}}, 0, \ldots, 0\right), & Q^{N_f} &= \left(\widehat{h}, 0\ldots, 0\right), \\ X &= \mathrm{diag}\,(x_h\sigma_3, \text{pseudoconf.})\,, & Y &= \mathrm{diag}\,(\widehat{y}_1\sigma_1 + y_h\sigma_3, \text{pseudoconf.})\,, \end{aligned} \tag{2.11}$$

where x_h is always a root of the meson deformation $m(x)$, while

$$y_h = -\frac{\alpha}{2\lambda x_h}\,, \qquad \lambda\widehat{y}_1^2 + V'(-x_h) + \frac{\alpha^2}{4\lambda x_h^2} = 0\,,$$

and the quark expectation values are $\widehat{\widetilde{h}}\widehat{h} = -\frac{1}{m_2}[V'(x_h) - V'(-x_h)]$. Again, the remaining diagonal expectation values for the adjoints in (2.11) are the generic pseudoconfining vacua (2.6) and (2.8) and we have to impose the tracelessness. The symmetry breaking pattern is as in (2.9) but now the sum of the ranks of the low energy SQCDs decreases by two, $\sum_{i=1}^{n+2} N_i + 2\sum_{i=1}^{\frac{n-1}{2}} \widehat{N}_i = N_c - 2$. In this nonabelian higgs case we higgs *two* color directions and this represents a new classical phase of SQCD.

2.3 D-terms

Consider the kinetic term for the adjoints

$$\int d^2\theta\, d^2\bar{\theta}\, \mathrm{Tr}\left(X^\dagger e^{adV} X + Y^\dagger e^{adV} Y\right),$$

the D-term equations of motion are $[X, X^\dagger] + [Y, Y^\dagger] = 0$. The abelian vacua (2.6), satisfy the D-term equation as usual. For the nonabelian vacua (2.8) and (2.11), however, due to the nonvanishing commutator of the Pauli matrices, we get the additional condition

$$\begin{array}{ll} \text{pseudoconf.} & \text{nonabelian higgs} \\ \mathrm{Im}\, cd^* = 0\,, & \mathrm{Im}\, \widehat{y}_1 y_h^* = 0\,. \end{array} \tag{2.12}$$

Note also that, if we set to zero the Lagrange multiplier α, then the term proportional to σ_3 in $\langle Y\rangle$ vanishes, so that the nonabelian vacuum automatically satisfies the D-term. This would amount to consider Y transforming in the adjoint of $U(N_c)$, rather than $\mathrm{SU}(N_c)$. In this way we would get rid of this additional D-term condition, since the vev that is subject to the constraint is proportional to α. However, if we compute the low energy matter content in the nonabelian vacua (2.8), we find that the $\operatorname{Tr} Y$, which is the U(1) part of the adjoint, becomes massless in this case. Albeit being neutral under the gauge interactions, the $\operatorname{Tr} Y$ field interacts with the other massive low energy degrees of freedom through superpotential terms. On the other hand, we need a mass gap in order to make sense of the glueball superpotential, so we are forced to keep the Lagrange multiplier α and the additional constraint (2.12) and we will consider the $\mathrm{SU}(N_c)$ gauge theory in the following.

2.4 The classical chiral ring

Consider the superpotential (2.1) and for simplicity drop all the lower relevant operators, keeping just $V'(x) = t_n x^n$. In this case the theory is superconformal and its flows have been studied in [11]. Using the equations of motion we get $((-)^n + 1)\, X^n Y = -2Y^3$, so that in the n odd case the chiral ring is truncated to $Y^3 = Y$ and is generated by the products $\operatorname{Tr} X^{k-1} Y^{j-1}$, for $k = 1, \dots, n$ and $j = 1, 2, 3$, regardless of the ordering. Due to $\{X, Y\} = -\frac{\alpha}{\lambda}$ and the cyclicity of the trace, the only nonvanishing chiral ring operators are actually

$$\begin{aligned} &\operatorname{Tr} X^{k-1}\,, \qquad k = 3, \dots, n, \qquad \operatorname{Tr} Y^2\,, \\ &\operatorname{Tr} X^{2k} Y^2\,, \qquad k = 1, \dots, \frac{1}{2}(n-1)\,, \end{aligned} \tag{2.13}$$

and also the dressed mesons $M_{kj} = \widetilde{Q} X^{k-1} Y^{j-1} Q$, for $k = 1, \dots, n$ and $j = 1, 2, 3$.

In the n even case, apparently, the chiral ring would not be truncated. We will consider just the n odd case in the following and get back to this issue in section 5.3, where we will show that indeed, by considering the flow from n odd to n' even, with $n' < n$, the chiral ring is truncated also in the even case.

We will be interested in solving for the expectation values of the operators of the chiral ring. We can collect them in four generating functions

$$\begin{aligned} Z(x,y) &= -\frac{1}{32\pi^2}\left\langle \operatorname{Tr} \frac{W_\alpha W^\alpha}{x - X}\frac{1}{y - Y}\right\rangle, \\ u_\alpha(x,y) &= \frac{1}{4\pi}\left\langle \operatorname{Tr} \frac{W_\alpha}{x - X}\frac{1}{y - Y}\right\rangle, \\ U(x,y) &= \left\langle \operatorname{Tr} \frac{1}{x - X}\frac{1}{y - Y}\right\rangle, \\ M^f_{\tilde f}(x,y) &= \left\langle \widetilde{Q}_{\tilde f}\frac{1}{x - X}\frac{1}{y - Y} Q^f\right\rangle. \end{aligned} \tag{2.14}$$

In a supersymmetric vacuum u_α must be vanishing, therefore we set it to zero. These loop functions (2.14) can be expanded in Laurent series of x or y, for instance the first one is

$$Z(x,y)=\sum_{k=0}^{\infty} x^{-1-k}R_k^Y(y)=\sum_{k=0}^{\infty} y^{-1-k}R_k^X(x)\,, \tag{2.15}$$

where we introduced the *generalized resolvents*

$$R_k^X(x)=-\frac{1}{32\pi^2}\left\langle \mathrm{Tr}\,\frac{W_\alpha W^\alpha}{x-X}Y^k\right\rangle, \tag{2.16}$$

and analogously for $R_k^Y(y)$. The leading term in the expansion (2.15) is the usual resolvent of the one-adjoint theory. It will be useful to introduce also a *generalized glueball* $\widetilde{S}=-\frac{1}{32\pi^2}\langle \mathrm{Tr}\,W_\alpha W^\alpha Y\rangle$. Since all the single trace operators of two adjoints can be extracted from $R_k^X(x)$, we can just solve for this operators and do not consider $R_k^Y(y)$. Analogous expressions to (2.16) hold for the other generalized resolvents

$$M_k(x)=\left\langle \widetilde{Q}\frac{1}{x-X}Y^kQ\right\rangle, \qquad T_k(x)=\left\langle \mathrm{Tr}\,\frac{1}{x-X}Y^k\right\rangle. \tag{2.17}$$

Let us consider the semiclassical expressions for the generators that we obtain by plugging into (2.14) the solutions for $\langle X\rangle$ and $\langle Y\rangle$ and for the fundamentals. Classically, the glueball vanishes, however we can keep it as a fixed parameter to study $Z(x,y)$. Then, we can Laurent expand it and obtain the semiclassical resolvent

$$R(x)=\sum_{i=1}^{n+2}\frac{S_i}{x-a_i}+2x\sum_{i=1}^{\frac{n-1}{2}}\frac{\widehat{S}_i}{x^2-\widehat{a}_i^2}\,, \tag{2.18}$$

which has poles at the classical vacua. The residues are S_i, for $i=1,\dots,n+2$, the glueballs for the SQCDs we flow to in the abelian vacua (2.6), and $\widehat{S}_i$, for $i=1,\dots,(n-1)/2$ the glueballs for the SQCDs we flow to in the nonabelian vacua (2.8).

The meson generator depends on the vacuum we consider. In the pseudoconfining phase $M(x,y)$ vanish, since the fundamentals vanish. In the abelian higgs (2.10) and nonabelian higgs phases (2.11), however, it is nonvanishing and we find

$$\text{abelian higgs} \qquad\qquad\qquad \text{nonabelian higgs}$$

$$M(x)=-\frac{1}{m'(x_h)}\frac{V'(x_h)+\frac{\alpha^2}{4\lambda x_h^2}}{x-x_h}\,, \qquad M(x)=-\frac{1}{m'(x_h)}\frac{V'(x_h)-V'(-x_h)}{x-x_h}\,, \tag{2.19}$$

The meson generator has poles at the higgs eigenvalues, whose residue depends on the couplings and the branch.[4]

3. The three-sheeted curve

In this section we will study the chiral ring in the quantum theory by making use of the Konishi anomaly equations. We will consider in details the curve of the gauge theory (2.1),

[4]We considered just the last flavor direction, i.e. $M(x)=M(x)_{N_f}^{N_f}$, according to the classical solutions (2.10) and (2.11).

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defined by the gauge theory resolvent $R(x)$. On the other hand, due to the DV correspondence, this curve provides the solution to the planar limit of the two-matrix model, whose action is given by the gauge theory superpotential (2.1). This matrix model has been solved by Ferrari [8] and we will collect its basic features in appendix A, in a gauge theory language.

In the following, we will study the semiclassical expansion of the resolvent and identify its analytic structure, i.e. the branch points. In appendix C we will work out the holomorphic differentials on the curve. Let us briefly recall that in the one-adjoint theory, described in [5], the curve is the hyperelliptic Riemann surface $w^2 = V'(x)^2 + \hbar f(x)$. Each of the two sheets corresponds to a classical phase of the gauge theory, that are the pseudoconfining and the higgs branches. The interpolation between the two branches is possible by continuously move the poles of the resolvents $M(x)$ and $T(x)$ through the two sheets. In our two adjoint theory, we have instead a cubic algebraic curve and the full quantum theory is described by a three-sheeted covering of the plane. We will see in the next sections that again each of the three sheets corresponds to one different branch, i.e. the pseudoconfining, the abelian higgs and the nonabelian higgs branches. This will confirm the proposal that the degree of the $\mathcal{N}=1$ curve corresponds to the number of semiclassical branches of the gauge theory. In the quantum theory we can interpolate between all the phases by moving poles around the curve.

3.1 The cubic equation

Unlike the one-adjoint theory [3], in this more general case there is some work to do in order to extract the algebraic equation of the curve from the anomaly equations. The strategy we will use is to derive some equations involving the generator $Z(x,y)$ in (2.14) and then, by considering the Laurent expansion of these equations, derive some recursion relations for $R_k(x)$ that magically close on the resolvent $R(x)$, which defines the algebraic curve. This procedure, proposed by Ferrari [8] in the corresponding matrix model, is reproduced as well on the gauge theory side. We summarize the basic Ward identities in appendix A.

The curve of the gauge theory is the following cubic

$$w^3 + a(x^2)w^2 + b(x^2)w + c(x^2) = 0\,, \tag{3.1}$$

where we introduced the auxiliary variable $w = x^2\,[R(x) - V'(x)]$. The coefficients are given by

$$\begin{cases} a(x^2) = x^2[V'(x)+V'(-x)] - \dfrac{\alpha^2}{4\lambda}, \\ b(x^2) = x^4 V'(x)V'(-x) - \dfrac{\alpha^2 x^2}{4\lambda}[V'(x)+V'(-x)] + \hbar x^2\left[x^2[F_0(x)+F_0(-x)] + \alpha\widetilde{S}\right], \\ c(x^2) = x^4\Big\{-\dfrac{\alpha^2}{4\lambda}V'(x)V'(-x) + \\ \qquad +\hbar\Big[x^2\big[F_0(-x)V'(x) + F_0(x)V'(-x) + 2\lambda\widetilde{F}_2(x^2) + \alpha\widetilde{F}_1(x^2)\big] + \\ \qquad +\frac{\alpha}{2}\widetilde{S}[V'(x)+V'(-x)]\Big] - \hbar^2\lambda\widetilde{S}^2\Big\}. \end{cases} \tag{3.2}$$

Let us clarify the notations. We have introduced the degree $n-1$ polynomials $F_k(x)$

$$F_k(x) \equiv -\frac{1}{32\pi^2}\left\langle \mathrm{Tr}\, W_\alpha W^\alpha \frac{V'(x)-V'(X)}{x-X} Y^k \right\rangle . \tag{3.3}$$

These are the generalization of the usual quantum deformation $f(x)$ in the one-adjoint hyperelliptic Riemann surface $w^2 = V'(x)^2 + f(x)$ and their coefficients are proportional to the glueballs, hence they vanish classically.[5] Moreover, we introduced the even combinations $2x^2\widetilde{F}_1(x^2) = x[F_1(x) - F_1(-x)]$ and $2\widetilde{F}_2(x^2) = F_2(x) + F_2(-x)$. Note that the coefficient of the leading term of the first polynomial $F_0(x)$ is the glueball $S = F_{0(n-1)}/t_n$.

The polynomial coefficients a, b, c of the curve (3.1) are even functions of x. The curve in fact is invariant under the automorphism $x \to -x$, that we will discuss below.

As explained in appendix A.4, it is convenient to shift $w \to w + a(x)/3$ to get rid of the subleading term and cast (3.1) to its normal form

$$f(w,x) = w^3 + 3\gamma(x^2)w + 2\delta(x^2) = 0\,, \tag{3.4}$$

where $\gamma(x^2)$ and $\delta(x^2)$ are the combinations $3\gamma = (b - \frac{a^2}{3})$ and $2\delta = (c - \frac{ab}{3} + \frac{2a^3}{27})$ of (3.2). Let us introduce the discriminant of the cubic equation $\Delta(x^2) = \gamma^3 + \delta^2$ and the auxiliary function $u(x) = (-\delta + \sqrt{\Delta})^{\frac{1}{3}}$. The general solutions to (3.1) is given by

$$\begin{aligned} w^{(I)} &= e^{i\frac{2}{3}\pi}u - e^{-i\frac{2}{3}\pi}\frac{\gamma}{u} - \frac{a}{3}\,, \\ w^{(II)} &= u - \frac{\gamma}{u} - \frac{a}{3}\,, \\ w^{(III)} &= e^{-i\frac{2}{3}\pi}u - e^{i\frac{2}{3}\pi}\frac{\gamma}{u} - \frac{a}{3}\,, \end{aligned} \tag{3.5}$$

and recalling the definition $w = x^2[R(x) - V'(x)]$ we find the three expressions for the resolvents

$$\hbar R^{(i)}(x) = V'(x) + \frac{w^{(i)}(x)}{x^2}\,, \qquad i = I, II, III\,.$$

The resolvent on the physical sheet will be identified as usual by its asymptotics $R(x) \sim S/x$. We can rearrange the asymptotic expansion as a semiclassical expansion in powers of $\hbar$ and find that the solution $w^{(I)}(x)$ has the correct physical behavior

$$R(x) = \frac{x^2F_0(x) + \frac{\alpha\widetilde{S}}{2}}{x^2V'(x) + \frac{\alpha^2}{4\lambda}} + \lambda x \frac{2\widetilde{F}_2(x^2) + \alpha\widetilde{F}_1(x^2)}{2v_-(x^2)\left[x^2V'(x) + \frac{\alpha^2}{4\lambda}\right]} + \mathcal{O}(\hbar)\,, \tag{3.6}$$

where $-2xv_-(x^2) = V'(x) - V'(-x)$ is the odd part of the adjoint polynomial and $S = F_{0(n-1)}/t_n$. The other two solutions in (3.5) describe the second and the third sheet, which are not visible classically, and we collect them in appendix A. Their leading term in the semiclassical expansion is

$$\hbar R^{II}(x) = V'(x) + \frac{\alpha^2}{4\lambda x^2} + \mathcal{O}(\hbar)\,,$$

[5]Our notations are slightly different from those of CDSW [3], i.e. $f(x) = -4F_0(x)$.

$$\hbar R^{III}(x) = V'(x) - V'(-x) + \mathcal{O}(\hbar)\,. \tag{3.7}$$

If we put $\hbar = 0$ in the anomaly equations, but still keeping the glueballs as parameters, $F_k(x) \neq 0$, we find as classical expression for the resolvent precisely the physical solution (3.6).

3.2 The branch points

Let us look at the analytic structure of the curve (3.4). The branch points are the singular points of the curve, that is the points at which $f = df = 0$. Since $\partial_w f = 3(w+\gamma)$, one can easily find that the singular points are given by the zeros of the discriminant $\Delta(x^2) = \gamma^3 + \delta^2$. The ramification index r_i of each of these branch points is such that $f(w,x)$ together with its $r_i - 1$ derivatives vanish at the point. This index tells how many sheets we can reach by winding around the branching point. The number of branch points would be $\deg \Delta = 6(n+2)$ where n is the degree of $V'(x)$ in the gauge theory. However, we can collect out an overall x^6 factor in front of Δ. Therefore, the number of branch points is $6(n+1)$. All of these branching points have ramification index $r_i = 2$ since $\partial_w^2 f(w,x) = 6w$ never vanishes at these points.

Let us look at the semiclassical expansion of the discriminant

$$-(27/x^6)\Delta = v_-(x^2)^2 p(x)^2 p(-x)^2 + \mathcal{O}(\hbar)\,. \tag{3.8}$$

Classically, we have $6(n+1)$ double zeros, which come in pairs symmetric under $x \to -x$. The first $n-1$ of them come from the zeros of $v_-(x^2)$, the odd part of the adjoint polynomial, and correspond to the nonabelian vacua (2.8). Other $n+2$ double zeros are given by the roots of $p(x)$ in (2.7) and correspond to the abelian vacua (2.6). The last $n+2$ double zeros are given by the roots of $p(-x)$ and are the reflection of the abelian vacua (2.7) under $x \to -x$. The latter are not classical vacua and we will shortly see how they arise. Consider now the image of $x = 0$. Even if Δ has an overall factor x^6, it turns out that $\partial_w f$ vanishes at $x = 0$ on the first and third sheet, but it is nonvanishing on the second sheet, so this is actually a cusp and not a branch point. Each double zero of the discriminant corresponds to a classical pole in the semiclassical expansion of the resolvent on some particular sheets.

In the quantum theory, each of these double zeros split up into two branch points. Since they all have branching number two, each branch point lies on two sheets of the algebraic curve. Therefore, to tell which sheets are connected by which branch point, we have to solve the conditions

$$w^{(i)}(a) = w^{(j)}(a)\,, \qquad i,j = I, II, III\,, \tag{3.9}$$

where $w^{(i)}$ are the three solutions (3.5). For instance, the points $x = a$ that lie on the first and second sheet satisfy $w^{(I)}(a) = w^{(II)}(a)$, which gives the condition $(-\delta(a))^{\frac{2}{3}} = e^{i\frac{\pi}{3}}\gamma(a)$. In practice, however, the expressions of $\gamma(a)$ and $\delta(a)$ are very complicated and involve the quantum deformations $F_k(x)$, but it turns out that it is sufficient to study the limit $\hbar = 0$, in which the cubic (3.1) factorizes into three disconnected sheets

$$\left[w - \frac{\alpha^2}{4\lambda}\right]\left[w^2 + x^2[V'(x) + V'(-x)]w + x^4 V'(x)V'(-x)\right] = 0\,, \tag{3.10}$$

whose solutions we can identify as the semiclassical limits of the resolvent on the three sheets

$$w_{cl}^{(I)}(x) = -x^2 V'(x)\,, \qquad w_{cl}^{(II)}(x) = \frac{\alpha^2}{4\lambda}\,, \qquad w_{cl}^{(III)}(x) = -x^2 V'(-x)\,. \tag{3.11}$$

In the exact solutions (3.5) they satisfy (3.9), which means that they are special points lying on two different sheets. In the classical limit, each couple of branch points degenerates into a pole located at the corresponding vacuum. We can solve the conditions (3.9) on the classical curve and identify which classical pole connects which sheets: on the curve (3.10) the vacua are represented by marked points on each disconnected sheets, such that the above conditions are satisfied. We find then

$$w_{cl}^{(I)}(a) = w_{cl}^{(II)}(a) \qquad \text{iff} \qquad a^2 V'(a) + \frac{\alpha^2}{4\lambda} = 0\,, \tag{3.12}$$

$$w_{cl}^{I}(a) = w_{cl}^{III}(a) \qquad \text{iff} \qquad v_-(a^2) = 0\,, \tag{3.13}$$

$$w_{cl}^{(II)}(a) = w_{cl}^{(III)}(a) \qquad \text{iff} \qquad a^2 V'(-a) + \frac{\alpha^2}{4\lambda} = 0\,. \tag{3.14}$$

The branch points (3.12) connecting the 1st with the 2nd sheet come from the splitting of the abelian vacua $x = a_i$, for $i = 1, \ldots, n+2$. We call I_i the $n+2$ branch cuts coming from the splitting of the abelian vacua at a_i. These branch cuts connect the 1st and 2nd sheet. The 1st and the 3rd sheet are connected by the branch points (3.13) coming from the splitting of the nonabelian vacua $x = \pm\widehat{a}_i$, for $i = 1, \ldots, (n-1)/2$. Note that for each nonabelian vacuum corresponding to a value of $\widehat{a}_i^2$, we have on the 1st sheet a pole in $\widehat{a}_i$ and another one in $-\widehat{a}_i$. We denote $\widehat{I}_i$ the branch cuts around $\widehat{a}_i$ and $\widehat{I}'_i$ the branch cuts around $-\widehat{a}_i$. These branch cuts connect the 1st and 3rd sheets. Finally, the 2nd and 3rd sheet are connected by the branch points (3.14) that split from $x = -a_i$, which are the reflections of the abelian vacua a_i. We will denote I'_i the branch cut around the pole at $-a_i$. These cuts connect the 2nd and 3rd sheet. Note that these cuts do not correspond to a gauge theory vacuum and, indeed, we cannot see them on the physical sheet. In this way we have accounted for all the $6(n+1)$ branch points. We can summarize the monodromy structure of the curve as follows

$$\begin{array}{cc} \text{Branch Cuts} & \text{Sheets} \\ I_i & I \leftrightarrow II \\ I'_i & II \leftrightarrow III \\ \widehat{I}_i, \widehat{I}'_i & I \leftrightarrow III \end{array} \tag{3.15}$$

We could have found the same results by looking at the semiclassical expression for the resolvent on the three sheets in (3.6) and in appendix A. In fact, the classical poles in the resolvents correspond precisely to the marked points on the three disconnected sheets. In the quantum theory, each marked point splits into two branch points.

The automorphism $x \to -x$ of the curve (3.1) exchanges the 1st and the 3rd sheets, while leaving the second sheet invariant. We do not discuss the noncompact B cycles.

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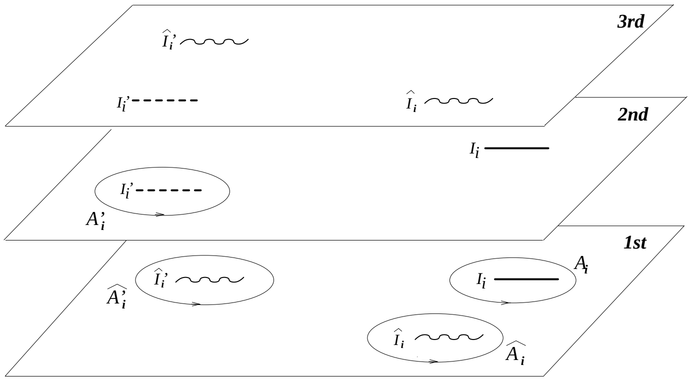

Figure 1: The three sheeted curve. The cut I_i comes from the splitting of the abelian vacua at $x = a_i$. The cuts $\widehat{I}_i$ and $\widehat{I}_i'$ come from the splitting of the nonabelian vacuum, at $x = \pm\widehat{a}_i$. The cut I_i' is not visible from the physical sheet. It comes from the splitting of the pole at $x = -a_i$. The A-periods are also shown.

For our purpose, in fact, we will always consider the gauge theory in the weak coupling expansion, so the periods on compact and noncompact cycle will never mix.

Now we can use the Hurwitz formula $2g - 2 = -2p + \sum_i (r_i - 1)$ to compute the genus g of the curve, where p is the number of sheets and the sum runs over all the branch points, $r_i = 2$ being the branching number of each. We find

$$g = 3(n+1) - 2\,. \tag{3.16}$$

In appendix C we compute the holomorphic differentials on the curve (3.1), showing that we have g of them.

3.3 The glueballs

In the one-adjoint theory [3], the usual way one defines the glueball S_i in the i-th low energy SQCD is by computing the period of the resolvent around the i-th cut on the physical sheet. In our case, the generic low energy SQCDs (2.9) come from abelian as well as nonabelian vacua and they require different definitions. In the case of the abelian vacua (2.6), we define the glueballs as usual

$$S_i = \oint_{A_i} R(x)\,, \tag{3.17}$$

where the A_i contour sorrounds the corresponding I_i abelian cut, see figure 1. This definition reproduces the semiclassical result (2.18) and is the same prescription as in [3]. On the other hand, the SQCD we flow to in the nonabelian vacuum is described on the physical sheet by the two cuts $\widehat{I}_i$ and $\widehat{I}_i'$, which are symmetric with respect to the origin. This phenomenon has been called "eigenvalue entanglement" in the related two-matrix model [8], where it was shown that the eigenvalue density $\rho(x)$ for such representations is symmetric, $\rho(x) = \rho(-x)$ for $x \in \widehat{I}_i \cup \widehat{I}_i'$. Since the gauge theory glueball corresponds to the matrix model filling fraction of the eigenvalues, the periods of the resolvent $R(x)$ around the cuts

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$\widehat{I}_i$ and $\widehat{I}'_i$ is the same. We define therefore the glueball as either period

$$\widehat{S}_i = \oint_{\widehat{A}_i} R(x) = \oint_{\widehat{A}'_i} R(x)\,. \tag{3.18}$$

This definition is consistent with the semiclassical resolvent (2.18), in fact we have that the total glueball is $S = \sum_{i=1}^{n+2} S_i + 2\sum_{i=1}^{\frac{n-1}{2}} \widehat{S}_i$ and is the residue of the resolvent at the pole at infinity. We will see below that this definition reproduces also the Konishi anomalies in these low energy SQCDs.

We would like to find that the number of glueballs corresponds to the number of parameters in the equation for the resolvent (3.1), which in turn is related to the genus of the curve. Recall first what happens in the one-adjoint theory with gauge group U(N) [3]. There, a degree n adjoint polynomial $V'(x)$ gives n low energy SQCD blocks with gauge group $U(N_i)$, each of which defines a glueball S_i [3]. The n glueballs S_i are in one to one correspondence to the n coefficients of the quantum deformation $f_{n-1}(x)$ of the $\mathcal{N}=1$ hyperelliptic curve $y^2 = V'(x)^2 + f_{n-1}(x)$. Finally, we can fix the coefficient of the leading term of $f(x)$ by the residue of the resolvent at infinity, due to the overall relation $\sum_i S_i = S$. The number of moduli of the curve is just the genus $g = n-1$, and the free parameters in $f(x)$ actually parameterize the moduli of the curve.

Now let us look at the cubic curve (3.1) and its coefficients (3.2) and identify the independent parameters. We have: the generalized glueball $\widetilde{S}$; the degree $n-1$ polynomial $F_0(x)$, with n coefficients; the polynomial $\widetilde{F}_2(x^2)$, which has $(n+1)/2$ coefficients; $\widetilde{F}_1(x^2)$ which has $(n-1)/2$ coefficients. However, by making use of the first anomaly equation in appendix A, one can show that the coefficients of $\widetilde{F}_1(x^2)$ can be recast as combinations of coefficients of $F_0(x)$, so they are not free parameters. We are left with a total of $1+n+(n+1)/2 = \frac{3}{2}(n+1)$ parameters, which is precisely the number of vacua, i.e. the low energy SQCDs in the generic vacuum (2.9). However, it might seem this is not in agreement with the number of independent deformations of the curve, which has genus $g = 3(n+1)-2$. But recall that the coefficients (3.2) of the curve are even functions, namely the curve has the automorphism $x \to -x$ that halves the number of moduli: this means that the periods of $R(x)$ around A_i and $\widehat{A}_i$ are respectively the same as those around A'_i and $\widehat{A}'_i$. Finally, the coefficient of the leading term of the quantum deformation $F_0(x)$ is fixed as the sum of all the glueballs, just as in the one-adjoint case we discussed above.

4. The mesons

We will solve now for the generator of the X-dressed mesons. Its classical expression depends on which of the three classical phases we are considering: it vanishes in the pseudoconfining phase, while it is given by (2.19) in the abelian and nonabelian higgs phase. Our strategy is again to consider a variation of the fundamentals and get an anomalous Ward identity.

In our case (2.1) the meson deformation is just X-dependent, $m(x) = m_1 + m_2 x$. Let us focus then on the X-dependent variation $\delta Q^f = Q^f \frac{1}{x-X}$, which gives the usual anomaly equation $[m(x)M(x)]_- = \hbar R(x)$, where we suppressed flavor indices. These considerations still hold if we consider the generalized meson generators $M_k(x) = \widetilde{Q}\frac{1}{x-X}Y^kQ$, whose anomaly equations are $[m(x)M_k(x)]_- = \hbar R_k(x)$. The explicit solution depends on the vacuum we are considering. We have three cases

1. **Pseudoconfining branch.** In this case the classical meson generator vanishes on the first sheet $M(x)|_{cl} = 0$, so we require that the spurious poles coming from the zeroes of $m(x)$ be cancelled

$$M(x) = \hbar\left(\frac{R(x)}{m(x)} - \frac{R^I(x_h)}{m'(x_h)}\frac{1}{x-x_h}\right). \tag{4.1}$$

2. **Abelian higgs branch.** This is characterized by the meson generator having a pole at the higgs eigenvalue x_h on the first sheet, whose residue is computed according to the first expression in (2.19). If we recall the expression of the resolvent on the second sheet in (3.7), we see that the boundary conditions are that the meson generator be regular on the second sheet

$$M(x) = \hbar\left(\frac{R(x)}{m(x)} - \frac{R^{II}(x_h)}{m'(x_h)}\frac{1}{x-x_h}\right). \tag{4.2}$$

This means that we can connect the branches (4.1) and (4.2) by moving the pole at x_h from the first to the second sheet by passing through one of the abelian cuts I_i.

3. **Nonabelian higgs branch.** This phase is characterized by the meson generator having a pole at x_h on the first sheet, whose residue is computed according to the second expression in (2.19). If we recall the expression of the resolvent on the third sheet in (3.7), we see that the boundary conditions in this case are that the meson generator be regular on the third sheet

$$M(x) = \hbar\left(\frac{R(x)}{m(x)} - \frac{R^{III}(x_h)}{m'(x_h)}\frac{1}{x-x_h}\right). \tag{4.3}$$

This means that we can connect the branches (4.2) and (4.3) by moving the pole at x_h from the second to the third sheet by passing through one of the cuts I'_i around $x = -a_i$, where a_i is one of the abelian pseudoconfining vacua. We can connect the nonabelian higgs solution to the pseudoconfining one by passing the pole from the third to the first sheet through one of the nonabelian cuts $\widehat{I}_i$.[6] We can summarize the solution for $M_k(x)$ in (2.17) as

$$M_k(x) = \frac{\hbar}{m(x)}\left(R_k(x) - R_k^{(i)}(x_h)\right), \tag{4.4}$$

[6]If the meson deformation only depends on the X adjoint, as in (2.1), then we can solve for the meson generator $M(x)$, but we can't write a closed equation for the generator of the Y-dressed mesons $M(y)$. In fact, by $\delta Q^f = Q^f\frac{1}{x-X}\frac{1}{y-Y}$ we get the anomaly equation $[m(x)M(x,y)]_- = \hbar Z(x,y)$, where $M(x,y)$ and $Z(x,y)$ are defined in (2.14). In the pseudoconfining branch, the mesons vanish classically, so we expect that the residues of $M(x,y)$ around the poles of $m(x)$ be vanishing in the classical limit. This gives $M(y) = -\hbar\sum_k Z(x_k,y)/m'(x_k)$. The same reasoning applies in the case the meson deformation only depends on Y instead, i.e. $\widetilde{Q}m(Y)Q$. Here, we can solve for the generator $M(y)$ but we can't get a nice expression for the generator $M(x)$. Eventually, if the meson deformation depends on both X and Y, then there is no easy way to study either meson generators.

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where $x_h = -m_1/m_2$ and the index $i = I, II, III$ labels respectively the pseudoconfining, abelian and nonabelian higgs branches and gives the resolvent on the three different sheets by (3.5). The explicit expressions for the higher $R_k(x)$ are given in appendix A. We can characterize the three branches as follows

branch	$M_k(x)$	
pseudoconfining	regular on I sheet	(4.5)
abelian higgs	regular on II sheet	
nonabelian higgs	regular on III sheet	

4.1 Konishi anomaly

Let us check that the mesons satisfy the Konishi anomaly in each low energy SQCD [4]. To get the expectation values of the meson operators in one low energy SQCDs, we just integrate the meson generator $M(x)$ around the corresponding cut. We have to distinguish whether the SQCD we flow to is in the abelian or in the nonabelian pseudoconfining vacuum. The i-th SQCD coming from the abelian vacuum (2.6) has N_f flavors and

$$\left\langle \widetilde{Q} X^j Q \right\rangle |_i = \oint_{A_i} x^j M(x)\,,$$

where we suppressed the flavor indices. The first meson gives the usual Konishi anomaly equation $\widetilde{Q}_f Q^f = \hbar N_f S_i / m(a_i)$, where $m(a_i)$ is the effective quarks mass in the i-th SQCD.

The j-th SQCD that comes from the nonabelian vacuum (2.8) gets twice as many flavors and requires some additional considerations. We can parameterize the $2N_f$ fundamentals as $(\widetilde{Q}_\alpha)^{\pm}_f$ and $(Q^\alpha)^{\pm,f}$ where $f = 1, \ldots, N_f$ is the flavor index and $\alpha = 1, \ldots, \widehat{N}_j$ is the color index and the additional index $\pm$ is another flavor index that comes from the splitting of the color indices in the nonabelian vacuum and the fact that the rank of the gauge group is halved in this vacuum. The superpotential of this SQCD is $W_{\rm eff} = m(\widehat{a}_j)\widetilde{Q}^+_f Q^{+,f} + m(-\widehat{a}_j)\widetilde{Q}^-_f Q^{-,f}$, so the two type of fundamentals Q^+ and Q^- have different mass. We have two different kind of mesons, which are decoupled. The $+$ mesons are given by the $\widehat{A}_j$ periods, the $-$ mesons by the $\widehat{A}'_j$ periods

$$\left\langle \widetilde{Q}^+ X^l Q^+ \right\rangle |_j = \oint_{\widehat{A}_j} x^l M(x)\,, \qquad \left\langle \widetilde{Q}^- X^l Q^- \right\rangle |_j = \oint_{\widehat{A}'_j} x^l M(x)\,. \tag{4.6}$$

If we recall the definition of the glueball in this vacuum (3.18), we find that the Konishi anomaly takes two expressions $\widetilde{Q}^+ Q^+ = \hbar N_f \widehat{S}_j / m(\widehat{a}_j)$ and $\widetilde{Q}^- Q^- = \hbar N_f \widehat{S}_j / m(-\widehat{a}_j)$.

5. Interpolating between the three branches

In this section we will solve for $T(x) = \mathrm{Tr}\,\frac{1}{x-X}$ and study its analytic behavior. We will find that it has only simple poles with integer residues and that the pseudoconfining, abelian higgs and nonabelian higgs branches of the gauge theory are described by three different configurations of the simple poles on the curve. As a result, the gauge theory curve is

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a degree three algebraic curve with marked points. Let us recall what happens in the one adjoint case [5]. In that case, the location of the poles of $T(x)$ characterizes either of the two branches of the theory: a pole on the physical sheet signals a semiclassical higgs branch, while when $T(x)$ is regular on the physical sheet we are in a semiclassical pseudoconfining branch. One can interpolate continuously between the two branches by moving the pole of $T(x)$ between the first and the second sheet through the branch cuts. In our two adjoint case, when $T(x)$ is regular on the physical sheet the theory is in the semiclassical pseudoconfining phase, while more complicated configurations describe the two higgs branches. Once again, we can reach all three branches by moving poles around the three sheets of the Riemann surface (3.1).

5.1 Solving for $T(x)$

To compute the exact expression of $T(x)$ is rather tedious. The strategy is analogous to the one we used for the resolvent $R(x)$, namely we collect some anomaly equations for $U(x, y)$, defined in (2.14), Then we consider its Laurent expansion and extract a linear equation for $T(x)$. The interested reader will find the details in appendix A. By the tree level superpotential (2.1) and the meson deformation $m(x) = m_1 + m_2\, x$ we obtain[7]

$$T(x) = \frac{N(x) + \delta N(x)}{D(x)}\,. \tag{5.1}$$

The notation is the following

$$\begin{aligned}
N(x) =& \, x^2 C_0(x) \left[V'(x) - V'(-x)\right] - \hbar R(x) \left[C_0(x) + C_0(-x)\right] - x^2 \left[2\lambda \widetilde{C}_2(x^2) + \alpha \widetilde{C}_1(x^2)\right], \\
D(x) =& \left[x^2 V(x) + \frac{\alpha^2}{4\lambda} - 2x^2 \hbar R(x)\right] \left[V'(x) - V'(-x) - 2\hbar R(x)\right] - \hbar^2 x^2 R(x)^2 + \\
& + \hbar x^2 \left[F_0(x) + F_0(-x)\right] + \hbar \alpha \widetilde{S}\,,
\end{aligned} \tag{5.2}$$

and we have introduced the degree $n-1$ polynomials

$$C_k(x) = \left\langle \mathrm{Tr}\, \frac{V'(x) - V'(X)}{x - X} Y^k \right\rangle, \tag{5.3}$$

which are analogous to (3.3), but do not vanish classically. In particular, the leading term of the first polynomial $C_0(x)$ is the rank of the unbroken gauge group $N_c = C_{0(n-1)}/t_n$. We have also used the combinations $2x^2 \widetilde{C}_1(x^2) = x[C_1(x) - C_1(-x)]$ and $2\widetilde{C}_2(x^2) = C_2(x) + C_2(-x)$. The term $\delta N(x)$ depends on the meson generators

$$\begin{aligned}
\delta N(x) \,=\, m_2 \Big(& -M(x)[V'(x) - V'(-x)] + \lambda [M_2(x) + M_2(-x)] + \\
& + \frac{\alpha}{2x} [M_1(x) - M_1(-x)] + \\
& + \hbar R(x) [M(x) + M(-x)]\Big),
\end{aligned} \tag{5.4}$$

[7] The only case one can solve the anomaly equations explicitly is when $m(x)$ is just linear in x as in (2.1). If higher Yukawa couplings are present this procedure does not work any more.

where $M_k(x) = \tilde{Q}\frac{1}{x-X}Y^k Q$ are the meson generators whose exact expression in the quantum theory is given in (4.4). Note that the term (5.4) vanishes if the superpotential does not have Yukawa couplings between the fundamentals and the adjoints. In the massive theory where $m(X) = const$ we would have just $T(x) = N(x)/D(x)$. The explicit expression of $\delta N(x)$ depends on the semiclassical branches through the meson generators.

Once we have the explicit solution (5.1) we can study its semiclassical expansion, its asymptotics and its singularities. The semiclassical expansion on the physical sheet is

$$T(x) = \frac{x^2 C_0(x)}{x^2 V'(x) + \frac{\alpha^2}{4\lambda}} + \lambda x \frac{2\widetilde{C}_2(x^2) + \alpha \widetilde{C}_1(x^2)}{2v_-(x^2)\left[x^2 V'(x) + \frac{\alpha^2}{4\lambda}\right]} + \hbar \frac{N_c S}{t_n x^{n+2}} + \mathcal{O}(\hbar^2)\,. \tag{5.5}$$

The first quantum correction has the asymptotics x^{-n-2}, so we keep it since it contributes to $\langle \mathrm{Tr}\, X^{n+1}\rangle$. The higher quantum corrections $\mathcal{O}(\hbar^2)$ begin at x^{2n+3}, so they do not contribute to the expectation values of the nontrivial operators in the chiral ring (2.13) and we can drop them. With some work one can compute also the asymptotics on the second and third sheet and find that the resolvent $T(x)$ has a simple pole at infinity on the first sheet with residue $-N_c$ and a simple pole at infinity on the third sheet with residue $N_c - 2$, where the -2 comes from $\delta N(x)/D(x)$. It is regular at infinity on the second sheet.

Let us find out the other poles of $T(x)$ on the various sheets in the three different branches. The only singularities of $T(x)$ are the ones at infinity and the simple poles at the images of the point x_h. The branches enter in the expression (5.1) of $T(x)$ only through the meson generators in (5.4). There is a nice pictorial way to see the three branches. Let us denote by a cross "$\times$" a simple pole with residue -1 and with a dot "$\bullet$" a simple pole with residue 1. We can summarize the singularities in the three branches as[8]

	pseudoconf.		abel. higgs		nonab. higgs	
	$-x_h$	x_h	$-x_h$	x_h	$-x_h$	x_h
III sheet	$\bullet$	$\bullet$	0	$\bullet$	0	0
II sheet	$\times$	$\bullet$	0	0	$\times$	$\bullet$
I sheet	0	0	0	$\bullet$	$\bullet$	$\bullet$

(5.6)

For each branch, in the first column we collect the residues at the images of $-x_h$ on the three sheets and in the second column the residues at the images of x_h. Note that, just as the meson generator in (4.5), each branch is characterized by the generator being regular on one of the three sheets. Therefore, we can label the second sheet the *abelian higgs sheet* and the third one the *nonabelian higgs sheet*. When the resolvents are regular on the physical sheet we are of course in the pseudoconfining phase, as shown in figure 2.

Let us consider the A-periods of T, recalling the definition of the glueballs in section 3.3. In the one-adjoint theory [3], the A-periods of T define the ranks of the i-th low energy

[8]The appearance of a pole with residue -1 for $T(x)$ might seem unexpected. Consider the semiclassical expansion of $T^{II}(x)$ in (5.1) on the 2nd sheet: $D(x)$, $N(x)$ and $\delta N(x)$ are all even functions of x as $T^{II}(x)$, which is regular at infinity. The contour integral of $T^{II}(x)$ on a large contour henceforth vanish. We can close the contour around the finite singularities, which are the poles at x_h and $-x_h$ and the periods around A_i and A'_i. We have $\sum_{i=1}^{n+2}\left(\oint_{A_i} + \oint_{A'_i}\right) T^{II}(x) = 0$ and $\left(\oint_{x_h} + \oint_{-x_h}\right) T^{II}(x) = 0$. The residue around $-x_h$ is thus the opposite of the residue around x_h.

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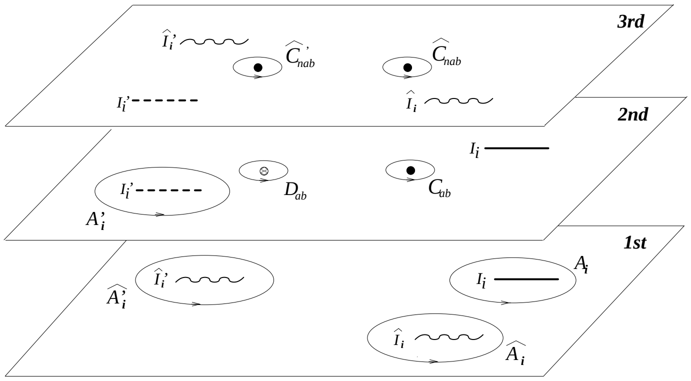

Figure 2: The pseudoconfining phase. The black and white dots represent poles for $T(x)$ with residue respectively one and minus one. The contours $\widehat{C}'_{nab}$ and D_{ab} enclose the images of the higgs eigenvalue $-x_h$, while $\widehat{C}_{nab}$ and C_{ab} enclose the images of x_h.

SQCD as $N_i = \oint_{A_i} T(x)$, but in this case we have two different kinds of SQCDs, by the flows in the abelian and nonabelian vacua. The A_i periods define the ranks of the SQCD in the abelian vacua (2.6) as usual

$$N_i = \oint_{A_i} T(x),$$

while the ranks of the SQCD in the nonabelian blocks (2.8) is computed by either periods around the nonabelian cuts

$$\widehat{N}_i = \oint_{\widehat{A}_i} T(x) = \oint_{\widehat{A}'_i} T(x)\,. \tag{5.7}$$

With these definitions we recover the residue of $T(x)$ at infinity in the physical sheet as the sum of the the ranks plus the higgs poles $N_c = \sum_{i=1}^{n+2} N_i + 2\sum_{i=1}^{\frac{n-1}{2}} + r_{ab} + 2r_{nab}$, where r_{ab} is 1 in the abelian higgs branch and vanishes otherwise, while r_{nab} is 1 in the nonabelian higgs branch and vanishes otherwise.

5.2 Interpolating between the three phases

Looking at the table (5.6), we can check that the sum of all residues of $T(x)$ on the curve vanishes. Moreover, when a cross meets a dot, they annihilate and, viceversa, from a vanishing residue we can create a pair cross-dot: $\times + \bullet = 0$. Now we can picture the way we interpolate between the three different branches as follows.

i) pseudoconfining $\leftrightarrow$ abelian higgs: Start with the pseudoconfining phase and move the dot $\bullet$ from the 2nd sheet to the 1st through the cut I_i. Due to the automorphism $x \to -x$, the other cross $\times$ in the second sheet moves through the symmetric cut I'_i from the 2nd to the 3rd sheet. Once on the 3rd sheet, the cross $\times$ annihilates with the $\bullet$, being both residues of a pole at $-x_h$, and we are left with the abelian higgs

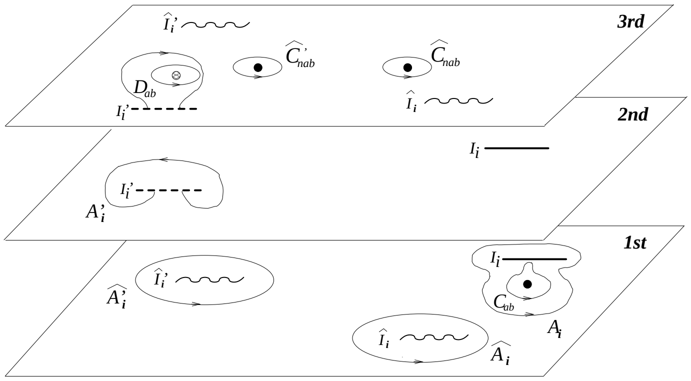

Figure 3: Interpolating between the pseudoconfining and the abelian higgs phase. Start in the pseudoconfining phase in figure 2 Then move the pole D_{ab} to the 3rd sheet through the cut I'_i and the pole C_{ab} to the 1st sheet through the cut I_i. On the 3rd sheet, the contours D_{ab} and $\widehat{C}'_{nab}$ combine giving vanishing residue for $T(x)$ at $-x_h$ on the 3rd sheet. The new period of $T(x)$ on the first sheet is around the contour $A_i|_{new} = A_i - C_{ab}$ and we find (5.8).

phase.

	pseudoconf.		$\longrightarrow$	abel.	higgs
	$-x_h$	x_h		$-x_h$	x_h
III sheet	$\bullet$	$\bullet$		$\times + \bullet = 0$	$\bullet$
II sheet	$\times \uparrow I'_i$	$\bullet \downarrow I_i$		0	0
I sheet	0	0		0	$\bullet$

When passing the pole through the i-th abelian cut I_i, the rank N_i of the corresponding i-th SQCD decreases by one. This is depicted in figure 3. The new contour in fact is $A_i|_{new} = A_i - C_{ab}$ and we find

$$N'_i = \oint_{A_i} T(x) - \oint_{C_{ab}} T(x) = N_i - 1\,. \tag{5.8}$$

ii) pseudoconfining $\leftrightarrow$ nonabelian higgs: Start with the pseudoconfining phase and move the two dots $\bullet$ from the 3rd sheet to the 1st through the nonabelian cuts: the pole at $-x_h$ moves through $\widehat{I}'_i$ and the pole at x_h moves through $\widehat{I}_i$.

	pseudoconf.		$\longrightarrow$	nonab. higgs	
	$-x_h$	x_h		$-x_h$	x_h
III sheet	$\bullet \uparrow \widehat{I}'_i$	$\bullet \uparrow \widehat{I}_i$		0	0
II sheet	$\times$	$\bullet$		$\times$	$\bullet$
I sheet	0	0		$\bullet$	$\bullet$

When passing the pole through the i-th nonabelian cut $\widehat{I}_i$, the rank $\widehat{N}_i$ of the corresponding i-th SQCD decreases by one. The new cycles are in fact $\widehat{A}_i|_{new} = \widehat{A}_i - \widehat{C}_{nab}$ and $\widehat{A}'_i|_{new} = \widehat{A}'_i - \widehat{C}'_{nab}$ and we find

$$\widehat{N}'_i = \oint_{\widehat{A}_i} T(x) - \oint_{\widehat{C}_{\rm nab}} T(x) = \widehat{N}_i - 1\,,$$

or equivalently for the other period around $\widehat{A}'_i$.

iii) nonabelian higgs $\leftrightarrow$ abelian higgs: Start with the nonabelian higgs phase, pass to the pseudoconfining phase by moving the poles from the physical sheet to the third one and then move to the abelian higgs phase.

5.3 Truncation of the chiral ring

The classical chiral ring of the gauge theory (2.1) is very different depending on whether $n = \deg V'(x)$ is odd or even. In the case $n =$ odd, we showed for $V'(x) = x^n$ that the independent operators are the ones in (2.13), and this still holds when deform the theory by lower relevant operators as for generic $V'(x)$. In particular, the chiral ring is characterized by the following relations, which are the anomaly equations that we derived in appendix A and we used to solve for $T(x)$

$$\begin{aligned} \operatorname{Tr} X^{k+1} Y + \frac{\alpha}{2\lambda} \operatorname{Tr} X^k &= 0\,, & (5.9)\\ \operatorname{Tr} X^l \left(V'(X) + \lambda Y^2\right) Y^k &= \mathcal{O}(\hbar)\,, & (5.10)\\ \operatorname{Tr} X^{2l+1} Y^{k+2} + \frac{\alpha}{2\lambda} \operatorname{Tr} X^{2l} Y^{k+1} &= \mathcal{O}(\hbar)\,, & (5.11) \end{aligned}$$

for $k, l \geq 0$. The first classical relation is exact, while the last two get quantum corrections. One can use these relations to prove that the operators $\operatorname{Tr} X^{2l+1} Y^2$, higher powers of $\operatorname{Tr} X^{2l \geq n} Y^2$, $\operatorname{Tr} Y^{j \geq 3}$ and $\operatorname{Tr} X^{j \geq n+1}$ can be expressed in terms of the basis (2.13).

If n' is even, on the other hand, the chiral ring is not truncated. This would mean that there are an infinite number of independent operators in the classical chiral ring. We want to address the n' even case. We start at the IR fixed point $\widehat{D}$ of the theory with superpotential $\operatorname{Tr} XY^2$, in the notations of [11]. Then we have two possibilities. If we consider the flow triggered by the relevant deformation $V'(X) = t_{n'} X^{n'}$ with $n' = 2m$, we can easily see that the chiral ring is not truncated. But we can consider the different flow, in the bottom line

$$\begin{array}{llll} \widehat{\mathbf{D}} \longrightarrow & \longrightarrow & \longrightarrow \mathbf{n}' = \text{even not trunc.} \\ \quad \searrow & & \\ & \mathbf{n} = \text{odd trunc.} & \longrightarrow \text{new } \mathbf{n}' = \text{even trunc.} \end{array} \qquad (5.12)$$

by first turning on a deformation such that $V'(X) = t_n X^n$ with $n = 2m+1$ and flow to the fixed point where the chiral ring is truncated. Then, we can switch on another relevant deformation such that $\delta V'(X) = t_{n'} X^{n'}$ with $n' = 2m < n$, that triggers a flow to another fixed point corresponding to the even case, but this fixed point is different from the untruncated one, namely here the chiral ring, inherited by the odd case, is still truncated. We can see this by considering the superpotential $V'(X) = t_n X^n + t_{n'} X^{n'}$. At the second fixed point, the first coupling t_n can be set to one, while the coupling $t_{n'}$ becomes marginal, so that actually $V'(X) = X^n + t_{n'} X^{n'}$.[9] Therefore, the chiral ring for n' even that we get

[9] The crucial point here is that if we started directly with the even coupling $t_{n'} X^{n'}$, we could not use (5.11) to eliminate $\operatorname{Tr} X^{n'} Y^2$. If we start with the odd coupling, on the contrary, we can do the job and then, when flowing to the even case, this equation is still valid, by just setting $t_n = 1$ and keeping the marginal coupling $t_{n'}$.

by flowing down from $n > n'$ contains the following operators[10]

$$\begin{aligned} &\operatorname{Tr} X^j\,, \qquad && j = 2, \dots, n'+1\,, \\ &\operatorname{Tr} X^{2i} Y^2\,, \qquad && i = 1, \dots, \frac{n'}{2}\,. \end{aligned} \tag{5.13}$$

We consider now the relations in the chiral ring of the theory with no fundamentals and we explain how the classical relations are modified in the quantum theory. Recall that in the one adjoint theory with superpotential $W = \operatorname{Tr} V(X)$, the classical chiral ring is described by the relation $\langle \operatorname{Tr} V'(X) \rangle = 0$. In the quantum theory, the anomaly equation $\langle \operatorname{Tr} \frac{V'(X)}{x-X} \rangle = 2\hbar R(x) T(x)$ tells us that classical relation still holds, but it gets modified by quantum corrections as long as we insert higher powers of X as $\left\langle \operatorname{Tr} X^k V'(X) \right\rangle = \mathcal{O}(\hbar)$ if $k \geq 1$.

Consider now the theory with two adjoints and superpotential (2.1). The classical chiral ring is described by the set of relations (5.9)–(5.11). The first and the third relations (5.9) and (5.11) correspond to the Y equation of motion with X insertions only and with both X and Y insertions. The second relation (5.10) corresponds to the X equation of motion with both X and Y insertions. With the help of the anomaly equations for the $T(x)$ resolvent, that we worked out in appendix A.3, we can compute the modifications that these relations are subject to in the quantum theory. The first relation (5.9) is exact. The second relation still holds for $l = 0$ and gets quantum corrections for $\lambda \geq 1$ and the third relation (5.11) gets quantum corrections for $k, l \geq 0$. Therefore we draw the general lesson that, when considering vacuum expectation values of gauge invariant chiral operators, the equations of motion a chiral superfield Φ still hold in the quantum theory, as long as we do not have additional insertions of Φ itself in the single trace correlator.

6. The classically invisible sheets and the branches

As anticipated in the Introduction, we would like to get some general understanding of the curve Σ of the $\mathcal{N} = 1$ supersymmetric gauge theory from this analysis of the different branches of SQCD. Consider a supersymmetric gauge theory with a matter content such that, once we fix the number of unbroken U(1)s in the low energy theory, there is no order parameter to distinguish between the various classical vacua in an invariant way, so have preferred the word *branches* rather than phases. This means that, even if the classical theory has different kinds of solution to the equations of motion, in the quantum theory we can reach all the different semiclassical behaviors, with the same number of unbroken U(1) gauge groups, by continuously moving the couplings along their moduli space, as in the case of a theory with matter in the fundamental representation.[11] It is clear that the different branches can only make sense in the limit of

[10]The analogous computation in the offshell chiral ring gives the following $\frac{3}{2}n + 3$ nontrivial operators $\operatorname{Tr} W_\alpha^2 X^j$ for $j = 0, \dots, n'+1$, $\operatorname{Tr} W_\alpha^2 Y$ and $\operatorname{Tr} W_\alpha^2 X^{2i} Y^2$ for $j = 1, \dots, \frac{n'}{2}$.

[11]Vacua with a different number of unbroken U(1)s, however, describe two different phases of the theory. In fact, as suggested in [5] and discussed in [13] for the one adjoint theory, if the i-th low energy SQCD has $N_i = 1$, it is not possible to further pass any pole through the corresponding cut in the onshell theory.

i) large expectation values, which is the semiclassical approximation;

ii) well separated branch cuts, that is far from singular points in the moduli space.

The observables that characterize the different branches are the ones in the chiral ring of the onshell theory. For instance, in the particular example of matter in the fundamental and adjoint representation, they are the resolvents $M(x)$ and $T(x)$, that we defined in (2.17). The semiclassical branches are then characterized by the analytic properties of these resolvents on the curve Σ, that is by their poles and the respective residues. We have proposed that an $\mathcal{N}=1$ gauge theory with a mass gap is described by a degree k algebraic curve, where k is the number of different branches of the theory. The curve is a k-sheeted covering of the plane, where each sheet corresponds to a different branch.[12]

In this way we can explain the appearance, in the quantum theory, of the "classically invisible sheets", to quote [5]. Let us see how this works and focus the attention on the meson operator $M(x)$. In our SQCD, the mesons are dressed by the adjoints, but with a more general matter content they would be dressed in some other ways and the general picture would not change. Each branch is characterized by a set of classical expectation values for the matter fields, which are set to the solutions to the equations of motion. In our case (2.1) for instance we have the pseudoconfining, abelian higgs and nonabelian higgs branches. Each branch is defined by the poles and residues of $M(x)$ on the first sheet at large expectation values (semiclassical regime). By the generalized Konishi anomaly equations, it follows that the generic form of $M(x)$ in the branch A is given by

$$M^A(x) = \hbar g(x) R(x) + q^A(x)\,, \tag{6.1}$$

where $g(x)$ is a rational function of the couplings, $R(x)$ is the gauge theory resolvent (which defines the curve) and $q^A(x)$ is another rational function that sets the boundary conditions on the meson operator, its poles and residues. Only in the classical limit do the branches make sense, thus we are interested in just the last term $q^A(x)$. In general, this depends on the resolvent $R(p_i)$ evaluated at the poles p_i, which are the images of the classical higgs expectation values. Since we assumed that there is no invariant way to distinguish the branches, we can connect all of them by changing continuously the boundary conditions $q^A(x)$, that is by moving the poles p_i between the sheets through the branch cuts. Now, since the resolvent $R(p_i)$ on the curve Σ gets as many different classical limits as the number of sheets (this is the way we identify the sheets), it turns out that, when taking the classical limit of large poles in the first sheet, we obtain as many different expressions as the number of sheets. Each one of them is a solution to the equations of motion and therefore a different branch. Suppose that we have k branches but $k+1$ sheets. Then, we can continue the pole p_i to that extra sheet and compute the classical limit for $M(x)$ on the first sheet, but this corresponds to a new solution to the equations of motion and so we have found a new branch.

For a generic $\mathcal{N}=1$ gauge theory, this holds with the following two caveats:

[12] We are excluding the case in which there is also a Coulomb branch, as it happens in the one adjoint theory (1.2) when $n=N$. In this case, for instance, the gauge group is broken to its Cartan subalgebra, there is no mass gap and, in the limit of vanishing superpotential, we recover the $\mathcal{N}=2$ SQCD.

- The number of sheets corresponds to the number of branches that we can distinguish in the effective description we are using. In our case of the deformed D_{n+2} theory (2.1), in the X effective description we can see only three branches, but we will argue below that new branches could be identified in the Y effective description.
- If there is an order parameter that characterizes one phase in an invariant way, then it seems plausible that the corresponding sheet be disconnected.

6.1 An example: the branches of SQCD

Let us see how our proposal works in the paradigmatic case of SQCD, where we have now a complete picture of all the possible branches. Depending on the extra matter content we can test our conjecture in different situations.

Ordinary SQCD: one sheet

Consider SQCD with gauge group $U(N_c)$. We can describe the offshell curve of this theory in a confining vacuum by adding a massive adjoint superfield X and integrating it out. The tree level superpotential is

$$W_{SQCD} = \frac{t_1}{2} \operatorname{Tr} X^2 + m\widetilde{Q}Q\,,$$

with $t_1 \gg m$. This theory classically has only one branch, the pseudoconfining one, in which both X and the fundamentals vanish. The corresponding curve is

$$y^2 = t_1^2 x^2 + 4\hbar t_1 S\,, \tag{6.2}$$

where S is the glueball. This looks like a double cover of the x plane with two branch points at $a^{\pm} = \pm 2\sqrt{S/t_1}$. But it is just a fake covering and the curve (6.2) actually describes the Riemann sphere. We have just one sheet corresponding to the one classical branch.

SQCD with one adjoint: two sheets

Consider $U(N_c)$ SQCD with one adjoint X and a confining phase superpotential

$$W = \operatorname{Tr} V(X) + \widetilde{Q}m(X)Q\,,$$

where $V'(x)$ has degree n and $m(x)$ degree $n-1$. This theory has received a lot of attention. For $n < N$, it has two branches, the pseudoconfining and the (abelian) higgs one. The curve is the well known hyperelliptic Riemann surface $y^2 = V'(x)^2 + \hbar f(x)$, where the degree $n-1$ polynomial $f(x)$ is the quantum deformation. For $n > 1$, this is a genuine double-sheeted covering of the x plane. As explained in [5], we can continuously interpolate between the pseudoconfining and higgs branch by moving the poles of $M(x)$ and $T(x)$ from the second to the first sheet. The first sheet corresponds to the pseudoconfining branch and the second to the higgs branch.

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SQCD with two adjoints: three sheets

This is the theory (2.1) that we have discussed at length. Classically, it has three branches: pseudoconfining (2.6)–(2.8), abelian higgs (2.10) and nonabelian higgs branch (2.11). The curve (3.1) is a three-sheeted covering of the plane and each sheet corresponds to a different branch, as explained in (5.6).

Up to now we have considered the effective description of two adjoint SQCD when integrating out the adjoint superfield X. However, we can as well integrate out the other adjoint Y and study the effective theory encoded in the resolvent

$$R(y) = -\frac{1}{32\pi^2}\left\langle \mathrm{Tr}\,\frac{W_\alpha W^\alpha}{y-Y}\right\rangle .$$

To find an algebraic equations for $R(y)$ we can use the same anomaly equations that we used for $R(x)$, but Laurent expand them in inverse powers of x instead, the details are in appendix A. One finds that $R(y)$ satisfies a degree 2^n algebraic equation [8]. The curve in the Y effective description is thus a 2^n sheeted covering of the plane. This might seem weird at first, since we conjectured that each sheet of the gauge theory curve is related to a different semiclassical phase. However, the three branches of our gauge theory are just the ones that we obtain by coupling the fundamentals with the adjoint X, and are the ones we can study in the X effective description of the theory. It would be nice to check that, when adding the most generic meson deformation $\delta W = \widetilde{Q}m(X,Y)Q$ to the superpotential, more complicated vacua appear corresponding to new higgs solutions, that we can just tell from each other in the effective Y description. On the X side they would be undistinguishable from the three branches we already considered.

7. The magnetic dual

In this section we will present some results on the Seiberg dual of the theory (1.4). A Seiberg dual description of the theory with gauge group $\mathrm{SU}(N_c)$ and superpotential

$$W_{el} = t_n\,\mathrm{Tr}\,X^{n+1} + \lambda\,\mathrm{Tr}\,XY^2 + \beta\,\mathrm{Tr}\,X\,, \tag{7.1}$$

has been proposed by Brodie in [9].[13] This theory flows to a nontrivial fixed point in the infrared. The magnetic dual is an $\mathcal{N}=1$ $\mathrm{SU}(3nN_f - N_c)$ gauge theory with two adjoint chiral superfields $\widetilde{X}$ and $\widetilde{Y}$, N_f magnetic fundamentals $\widetilde{q}^{\tilde{f}}$ and q_f and $3n$ gauge singlets $(P_{l,j})^f_{\tilde{f}}$ for $l = 1,\dots,n$ and $j = 1,2,3$, each of which transforms in the $(N_f, \bar{N}_f)$ of the flavor symmetry group. A magnetic superpotential was proposed

$$W_m = \bar{t}_n\,\mathrm{Tr}\,\widetilde{X}^{n+1} + \bar{\lambda}\,\mathrm{Tr}\,\widetilde{X}\widetilde{Y}^2 + \widetilde{q}\widetilde{m}(X,Y)q + \widetilde{\beta}\,\mathrm{Tr}\,\widetilde{X}\,, \tag{7.2}$$

where $\bar{t}_n = -t_n$ and $\bar{\lambda} = -\lambda$ and $\widetilde{m}(X,Y)$ is a polynomial that couples the magnetic fundamentals to the gauge singlets and the adjoints. This represents a Legendre transform

[13] This theory has no Lagrange multiplier for Y. Setting $\alpha = 0$ means that the adjoint Y gains an overall nonvanishing $\mathrm{U}(1)$ component $\mathrm{Tr}\,Y$.

between the electric and magnetic mesons. As opposed to our theory with confining phase superpotential (2.1), the theory (7.1) has just $n+2$ one-dimensional vacua and no two-dimensional vacua is present in this case.

We are interested in finding its generalization when we deform it by the confining phase superpotential

$$W_{el} = \mathrm{Tr}\, V(X) + \lambda\, \mathrm{Tr}\, XY^2 + \beta\, \mathrm{Tr}\, X + \alpha\, \mathrm{Tr}\, Y\,, \tag{7.3}$$

allowing for abelian as well as and nonabelian vacua (the latter are not present in Brodie's case).[14] The magnetic tree level superpotential is this case is not just the analogue of (1.4), but contains an extra term

$$W_{mag} = \mathrm{Tr}\, \bar{V}(\widetilde{X}) + \bar{\lambda}\, \mathrm{Tr}\, \widetilde{X}\widetilde{Y}^2 + \bar{s}\, \mathrm{Tr}\, \widetilde{Y}^2 + \widetilde{\beta}\, \mathrm{Tr}\, \widetilde{X} + \bar{\alpha}\, \mathrm{Tr}\, \widetilde{Y} + \widetilde{q}\widetilde{m}(X,Y)q\,, \tag{7.4}$$

corresponding to the extra coupling $\bar{s}$, which is fixed by duality to $\bar{s} = \lambda \frac{t_{n-1}}{t_n}\frac{N_f}{N_c}$. Unfortunately, for generic n one cannot solve the anomaly equations in the magnetic theory on a closed set of resolvents, due to this extra $\bar{s}$ coupling, so we have to stick to the classical duality map. However, in the D_3 case we can integrate out one of the two adjoints and flow to the well known one adjoint duality, which was studied by Kutasov, Schwimmer and Seiberg [14]. In this case, we will be able to give the duality map in the quantum theory between the electric and magnetic couplings and quantum deformation. As a consistency check, we reobtain the duality map which was found in [10].

Let us make a brief digression about the magnetic polynomial $\widetilde{m}(x,y)$. The *rationale* behind this Legendre transform term is that it must decouple the electric and magnetic mesons in different low energy SQCD blocks. This superpotential term can be conveniently parameterized by the kernel

$$\widetilde{m}(\widetilde{X},\widetilde{Y}) = \frac{1}{\mu^4}\oint dz\, dw\, \frac{\bar{V}'(z) - \bar{V}'(\widetilde{X})}{z-\widetilde{X}}\, \frac{w(\bar{\lambda}w^2+\widetilde{\beta}) - \widetilde{Y}(\bar{\lambda}\widetilde{Y}^2+\widetilde{\beta})}{w-\widetilde{Y}} P(z,w)\,, \tag{7.5}$$

that reproduces the one-adjoint term of [14, 10] in the case $n=1$. Note that, for $n>1$, (7.5) works only in the case $v_+(x^2)=0$ and for $\alpha=0$. In (7.5) we collected the $3n$ gauge singlets $(P_j)^{f}_{\tilde{f}}$ into a single meromorphic function

$$P(z,w) = \frac{P^{(1)}(z)}{w} + \frac{P^{(2)}(z)}{w^2} + \frac{P^{(3)}(z)}{w^3}\,, \qquad P^{(j)}(z) = \sum_{l=1}^{n} \frac{P^{(j)}_l}{z^l}\,. \tag{7.6}$$

For nonvanishing α, (7.5) does not decouple all the low energy SQCD blocks. The difficulty in finding such polynomial in this generic case might be related to the fact that, as we stressed in section 2.3, we have to impose also the D-term equations of motion (2.12) on the pseudoconfining vacua.

Consider the vacua of the magnetic theory (7.4), they are very similar to the electric ones (2.6) and (2.8). The equations of motion are

$$\bar{V}'(\widetilde{X}) + \widetilde{\beta} + \bar{\lambda}Y^2 = 0\,, \qquad \{\bar{\lambda}\widetilde{X} + \bar{s}, \widetilde{Y}\} + \widetilde{\alpha} = 0\,, \tag{7.7}$$

[14]In this section we will explicitly keep track of the multiplier β by displaying it separately from the adjoint polynomial $V'(x) = \sum_{i=1}^{n} t_i x^i$ on the electric as well as on the magnetic side.

and their irreps are still one-dimensional vacua and two-dimensional vacua. The $n+2$ abelian vacua are analogous to (2.6) but with magnetic eigenvalues and multiplicities instead, such that (2.7) is replaced by $\bar{p}(x) = \left(x^2 + \frac{\bar{s}}{\lambda}\right)[\bar{V}'(x) + \widetilde{\beta}] + \frac{\widetilde{\alpha}^2}{4\lambda}$ and $\bar{b}_i = -\frac{\widetilde{\alpha}}{2\lambda(\bar{a}_i + \frac{\bar{s}}{\lambda})}$. The $(n-1)/2$ nonabelian vacua are analogous to (2.8), but each block of the adjoint is replaced by $\widetilde{X} = \widehat{\bar{a}}_i\sigma_3 - \frac{\bar{s}}{\lambda}\mathbb{1}_2$ and $\widetilde{Y} = \bar{d}_i\sigma_3 + \bar{c}_i\sigma_1$ and $\widehat{\bar{a}}_i, \bar{d}_i, \bar{c}_i$ are fixed by (7.7).

We can use the SQCD duality relation $\bar{N}_i = \#\,\text{flavors} - N_i$ in each low energy SQCD block to check that the ranks of the gauge groups match. The electric low energy theory we flow to in the $n+2$ abelian vacua is SQCD with N_f flavors, while in the $\frac{n-1}{2}$ nonabelian vacua it is SQCD with $2N_f$ flavors. Therefore we have

$$\bar{N}_c = \sum_{i=1}^{n+2} \bar{N}_i + 2\sum_{i=1}^{\frac{n-1}{2}} \widehat{\bar{N}}_i = \sum_{i=1}^{n+2}(N_f - N_i) + 2\sum_{i=1}^{\frac{n-1}{2}}(2N_f - \widehat{N}_i) = 3nN_f - N_c\,.$$

Now we want to find the classical duality map. We will use a strategy analogous to KSS [14]. Let us first try a naive map between electric and magnetic eigenvalues $\bar{a}_i = a_i$ and $\bar{b}_i = b_i$.[15] We have to impose the tracelessness condition on both the electric and magnetic adjoint vacua. On the electric side, the nonabelian vacua (2.8) are already traceless and only the abelian vacua contribute, fixing the Lagrange multipliers β and α. On the magnetic side, the condition $\operatorname{Tr}\widetilde{Y} = 0$ gives $\sum_{i=1}^{n+2}\frac{1}{a_i} = 0$, but this is automatically satisfied because the linear term in the abelian polynomial (2.7) vanishes. Then we have to impose $\operatorname{Tr}\widetilde{X} = 0$, where also a contribution from the nonabelian magnetic vacua appear. Since a_i are the roots of (2.7) we have $\sum_{i=1}^{n+2} a_i = -\frac{t_{n-1}}{t_n}$ and the tracelessness condition can not be satisfied, unless $t_{n-1} = \bar{s} = 0$. If t_{n-1} vanishes, then the trivial map works.

7.1 The shift of the electric and magnetic theory

To find the map for nonvanishing t_{n-1} we follow the usual trick in singularity theory and shift the electric and magnetic adjoints. The new feature is that we need to add the new coupling $\operatorname{Tr}\widetilde{Y}^2$ to the magnetic side. Then we impose the tracelessness conditions on the shifted adjoints and find that the naive map works in the shifted variables.

Consider the electric theory (7.3) and shift X as $X = X_s - B\mathbb{1}\text{n}$. We do not shift Y since duality would fix the Y shift to zero. Then the electric superpotential reads

$$W_{el} = \operatorname{Tr} V_s(X_s) + \beta_s\left(\operatorname{Tr} X_s - BN_c\right) + \lambda \operatorname{Tr} X_s Y^2 + \alpha \operatorname{Tr} Y - \lambda B \operatorname{Tr} Y^2 + \phi N_c\,, \qquad (7.8)$$

where

$$V_s(X_s) = \sum_{i=1}^{n}\frac{g_i}{i+1}X_s^{i+1}\,, \qquad g_l = \sum_{i=l}^{n}\binom{i}{l}t_i(-B)^{i-l}\,,$$
$$\beta_s = \beta + \sum_{i=1}^{n} t_i(-B)^i\,, \qquad \phi = \sum_{i=1}^{n}\frac{i+2}{i+1}t_i(-B)^{i+1}\,. \qquad (7.9)$$

[15] Since we will stick to the electric pseudoconfining phase, we can forget about the magnetic polynomial $\widetilde{m}(X,Y)$ for the moment.

The shifted equations of motion are $V_s'(X_s)+\beta_s+\lambda Y^2=0$ and $\lambda\{X_s-B,Y\}+\alpha=0$. Their irreps are still abelian and nonabelian pseudoconfining vacua. Then, by imposing the tracelessness conditions $\mathrm{Tr}\,\widetilde{X}_s=\mathrm{Tr}\,Y=0$ we fix the electric Lagrange multipliers.

On the magnetic side we have to consider the magnetic superpotential (7.4) with the extra deformation $\delta\bar{W}=\bar{s}\,\mathrm{Tr}\,\widetilde{Y}^2$. At the end this new coupling will be fixed by duality as a function of the other couplings. Let us shift the magnetic theory as $\widetilde{X}=\widetilde{X}_s-\bar{B}\mathbb{1}$, the superpotential reads

$$\bar{W}_m=\mathrm{Tr}\,\bar{V}_s(\widetilde{X}_s)+\widetilde{\beta}_s\left(\mathrm{Tr}\,\widetilde{X}_s-\bar{B}\bar{N}_c\right)+\bar{\lambda}\,\mathrm{Tr}\,\widetilde{X}_s\widetilde{Y}^2+\widetilde{\alpha}\,\mathrm{Tr}\,\widetilde{Y}+(\bar{s}-\bar{\lambda}\bar{B})\,\mathrm{Tr}\,\widetilde{Y}^2+\bar{\phi}\bar{N}_c+f_s\,,$$

where the notation is as in (7.9) but with magnetic quantities instead and $f_s(t_l,\lambda,B)$ is the shifted coupling dependent function. If we solve the equations of motion we still find the abelian and nonabelian vacua and, by imposing again that the magnetic adjoints be traceless we find

$$\mathrm{Tr}\,\widetilde{X}_s=\sum_{i=1}^{n+2}(N_f-N_i)(\bar{a}_{s,i})+2\left(\bar{B}-\frac{\bar{s}}{\bar{\lambda}}\right)\sum_{i=1}^{\frac{n-1}{2}}(2N_f-\widehat{N}_i)=0\,, \tag{7.10}$$

where $\bar{a}_{s,i}$ are the shifted magnetic abelian vacua. Once we impose both electric and magnetic tracelessness conditions, we can postulate the naive match of the shifted eigenvalues

$$\bar{a}_{s,i}=a_{s,i}\,, \tag{7.11}$$

and we crucially drop the dependence upon the vacua inside (7.10) by fixing the shifts

$$\begin{aligned} B&=\frac{g_{n-1}}{2(n-1)g_n}\,,\\ \bar{B}&=B+\frac{\bar{s}}{\bar{\lambda}}\,. \end{aligned} \tag{7.12}$$

By comparing the electric and magnetic abelian vacua we get the map between the shifted couplings and Lagrange multipliers

$$\begin{aligned} \bar{g}_l&=-g_l\,, & \bar{\lambda}&=-\lambda\,,\\ \bar{\beta}_s&=-\beta_s\,, & \bar{\alpha}&=-\alpha\,. \end{aligned} \tag{7.13}$$

A sufficient condition for the map between the operators to be independent of the vacua is that the electric and magnetic superpotential match and the coupling dependent function $f_s(t_l,\lambda,B)$ do not depend on the vacuum. One can easily check this last requirement by differentiating the effective action with respect to the shifted couplings $\mathrm{Tr}\,X_s^{l+1}=-\,\mathrm{Tr}\,\tilde{X}_s^{l+1}+(l+1)\frac{\partial f_s}{\partial g_l}$ and see that $\partial_{g_l}f_s$ does not depend on $N_i,\widehat{N}_i$.

7.2 The map in the original couplings

By using (7.11), (7.12) and the map (7.13) we can reobtain the relation between the eigenvalues and the couplings in the original parametrization of the theory

$$\bar{a}_i=a_i-\frac{\bar{s}}{\bar{\lambda}},$$

$$\bar{t}_l = -\sum_{i=l}^{n} \binom{i}{l} t_l \left(\frac{\bar{s}}{\bar{\bar{\lambda}}}\right)^{i-l} . \tag{7.14}$$

While the abelian X eigenvalues are shifted by the duality, the nonabelian eigenvalues match exactly, as well as the Y expectation values

$$\begin{aligned} X: \quad & \widehat{\bar{a}}_i = \widehat{a}_i , \\ Y: \quad & \bar{b}_i = b_i , \qquad \bar{c}_i = c_i , \qquad \bar{d}_i = d_i . \end{aligned} \tag{7.15}$$

By imposing the tracelessness condition in the unshifted magnetic adjoint we fix $\bar{s}$ in terms of the other couplings of the theory

$$\bar{s} = \lambda \frac{t_{n-1}}{t_n} \frac{N_f}{\overline{\overline{N}}_c}.$$

In other words, we can write down the electric superpotential as (7.3), while the magnetic one (7.4) reads in electric variables

$$W_m = -\operatorname{Tr} V(\tilde{X} + \frac{\bar{s}}{\bar{\bar{\lambda}}}) - \lambda \operatorname{Tr}(\tilde{X} + \frac{\bar{s}}{\bar{\bar{\lambda}}}) t Y^2 - \alpha \operatorname{Tr} \widetilde{Y} - \beta \operatorname{Tr} \widetilde{X} + f(\text{coupl.}) . \tag{7.16}$$

7.3 Duality for D_3: the quantum theory

We would like to solve for the chiral ring operator (2.14) both in the electric and magnetic side in the quantum theory and find the map between the dual quantum deformations $F_k(x)$. For generic superpotentials (7.3) and (7.4) this seems impossible, since on the magnetic side the anomaly equations do not close any more on an algebraic equation for the magnetic resolvent $\widetilde{R}(x)$, due to the extra coupling $\operatorname{Tr} \widetilde{Y}^2$. The only case in which we can solve both electric and magnetic quantum theories is when the adjoint polynomial is just a mass term

$$W_{el} = \operatorname{Tr} \left(\frac{t_1}{2} X^2 + \beta X + \lambda X Y^2 + \alpha Y \right) + m \widetilde{Q} Q . \tag{7.17}$$

We will denote (7.17) the D_3 theory. The duality map in this case will reproduce exactly the KSS duality in the case of a one-adjoint SQCD [14, 10], that is we will see that $D_3 \sim A_3$, as expected from singularity theory. The D_3 theory have the $n+2=3$ abelian vacua (2.6) but no nonabelian vacua (2.8), which are only present if $n \geq 3$. Here X is massive and we can integrate it out upon its equations of motion, obtaining, at energies below the mass scale t_1, an effective superpotential $U_{\text{eff}}(Y) = -\frac{1}{2t_1} \operatorname{Tr}(\beta + \lambda Y^2)^2 + \alpha \operatorname{Tr} Y$, whose derivative is

$$U'(y) = -\frac{2\lambda}{t_1}(\beta y + \lambda y^3) + \alpha . \tag{7.18}$$

In the quantum theory, the anomaly equation for the resolvent $R^Y(y)$ reduces to the hyperelliptic curve

$$\hbar^2 R(y)^2 - U'(y) R(y) - \frac{1}{4} f(y) = 0 . \tag{7.19}$$

This is the usual anomaly equation for the one matrix model [3], where $U'(y)$ is the effective superpotential (7.18) and

$$f(y) = \frac{8\lambda}{t_1^2}(\beta F_0 + \lambda F_2 + \lambda y t_1 \widetilde{S} + \lambda y^2 F_0)\,, \tag{7.20}$$

is the quantum deformation, that we expressed in terms of the parameters that we used in the solution for $R(x)$ in (3.2).[16] The solution of the anomaly equation (7.19) is the well known $2\hbar R(y) = U'(y) - \sqrt{U'(y)^2 + \hbar f(y)}$. The physical picture in this case is the following. Classically, the resolvent $R(y)$ has three poles located at the classical vacua $y = b_i$, where the b_i's are given in (2.6). These are the roots of the cubic effective polynomial $U'(y)$. In the quantum theory, each pole splits into two branch points, that connect the first and the second sheet of the hyperelliptic curve. In this case we just have two classical branches, the pseudoconfining and the abelian higgs vacua. The nonabelian higgs phase is only present if $n \geq 3$. So we just have two sheets as in the one-adjoint theory. We can get the expectation values of the dressed mesons in the quantum theory by computing the contour integrals of the meson generator $M(y) = \widetilde{Q}\frac{1}{y-Y}Q = m^{-1}R(y)$ and find

$$\widetilde{Q}Q = \frac{S}{m}, \qquad \widetilde{Q}YQ = \frac{\widetilde{S}}{m}, \qquad \widetilde{Q}Y^2Q = \frac{F_2}{t_1}. \tag{7.21}$$

The magnetic theory dual to (7.17) is

$$W_{mag} = \frac{\bar{t}_1}{2}\operatorname{Tr}\widetilde{X}^2 + \bar{\lambda}\operatorname{Tr}\widetilde{X}\widetilde{Y}^2 + \bar{\beta}\operatorname{Tr}\widetilde{X} + \bar{\alpha}\operatorname{Tr}\widetilde{Y} + \widetilde{q}\widetilde{m}(X,Y)q + \bar{m}\operatorname{tr}P_1^{(1)}\,,$$

where the last term corresponds to the electric mass term for the fundamentals and the trace is over flavor indices. The magnetic polynomial $\widetilde{m}(X,Y)$ in this case is (7.5), even if we keep a nonvanishing α multiplier. Actually, the polynomial $\widetilde{m}(X,Y)$ is equivalent to what we get by using the effective quartic one-adjoint polynomial $\bar{U}(\widetilde{Y})$ in (7.18). In fact, the $\widetilde{X}$ dependence drops and we are left with

$$\widetilde{m}(X,Y) = \frac{\bar{t}_1}{\mu^4}\oint dw \frac{\bar{U}'(w) - \bar{U}'(\widetilde{Y})}{w - \widetilde{Y}}\,, \tag{7.22}$$

The magnetic singlet equations of motion are

$$\widetilde{q}Y^j q = -\delta_{j,2}\frac{\bar{m}\mu^4}{\bar{\lambda}\bar{t}_1}\,. \tag{7.23}$$

The anomaly equations for the resolvents $\widetilde{R}^{\widetilde{X}}(x)$ and $\widetilde{R}^{\widetilde{Y}}(y)$ are the same as in the electric theory, (3.1) and (7.19). We want to solve for the singlets in the magnetic theory, following the method in [10]. The magnetic meson generator $\widetilde{M}(y) = \widetilde{q}\frac{1}{y-\widetilde{Y}}q$ satisfies the anomaly equation $[\widetilde{m}(y)\widetilde{M}(y)]_- = \widetilde{R}^{\widetilde{Y}}(y)$. The generic solution is

$$\widetilde{M}(y) = \frac{\widetilde{R}^{\widetilde{Y}}(y)}{\widetilde{m}(y)} + \frac{\widetilde{r}(y)}{\widetilde{m}(y)}\,. \tag{7.24}$$

[16]From the algebraic equation in appendix A we get $f(y) = \frac{8\lambda}{t_1}(\lambda y^2 S + \lambda y\widetilde{S} - t_1\widehat{S}_1) = 0$, where $\widehat{S}_1 = -\frac{1}{32\pi^2}\langle\operatorname{Tr}W_\alpha W^\alpha X\rangle$. By using the recursion relations in appendix A we can also express the parameter $\widehat{S}_1$ in terms of the parameters $F_0 = t_n S, F_2$ that we used in the solution for $R(x)$ in (3.2) as $t_1\widehat{S}_1 = -\frac{2\lambda}{t_1}F_2 - \beta S$.

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The way we solve it is by first fixing $\widetilde{r}(y)$ to cancel spurious singularities from the zeros of $\widetilde{m}(y)$ and then solving for $\widetilde{m}(y)$ such that the singlet equations of motion (7.23) are satisfied. The solution in the pseudoconfining phase is $\widetilde{r}(y) = 0$ and

$$\widetilde{m}(y) = -\frac{\bar{t}_1^2}{\bar{\lambda}} \frac{\widetilde{f}(y)}{8\bar{m}\mu^4}\,, \tag{7.25}$$

where $\widetilde{f}(y) = \frac{8\bar{\lambda}}{\bar{t}_1^2}(\widetilde{\beta}\widetilde{F}_0 + \bar{\lambda}\widetilde{F}_2 + \bar{\lambda}y\bar{t}_1\widetilde{\bar{S}} + \bar{\lambda}y^2\widetilde{F}_0)$ is the quantum deformation of $\widetilde{R}^{\widetilde{Y}}(y)$ and $\widetilde{F}_k$ are the quantum deformations of $\widetilde{R}^{\widetilde{X}}(x)$, the magnetic version of (3.2). Formally it is the same expression as in (7.20), but replacing electric with magnetic quantities.

The magnetic polynomial in terms of the singlets reads $\widetilde{m}(y) = \frac{\bar{t}_1}{\mu^4}(\bar{\lambda}P_1^{(3)} + \bar{\lambda}yP_1^{(2)} + (\widetilde{\beta} + \bar{\lambda}y^2)P_1^{(1)})$. We can read off the expression of the quantum expectation value of the gauge singlets in terms of the quantum deformations

$$P_1^{(1)} = -\frac{\widetilde{F}_0}{\bar{m}\bar{t}_1}, \qquad P_1^{(2)} = -\frac{\widetilde{\bar{S}}}{\bar{m}}, \qquad P^{(3)} = -\frac{\widetilde{F}_2}{\bar{m}\bar{t}_1}\,. \tag{7.26}$$

If we match them directly to the electric mesons (7.21) we find the map duality map in the quantum theory between the quantum deformations $\widetilde{F}_0 = F_0$ and $\widetilde{F}_2 = F_2$ and then between the couplings, the Lagrange multipliers and the glueballs

$$\begin{aligned} \bar{t}_1 &= -t_1\,, \qquad \bar{\lambda} = -\lambda\,, \qquad \widetilde{\beta} = -\beta\,, \\ \bar{S} &= -S, \qquad \widetilde{\bar{S}} = -\widetilde{S}\,. \end{aligned} \tag{7.27}$$

8. Further directions

In this paper we have proposed a general explanation of the presence of the classically "invisible" sheets in the curves of $\mathcal{N} = 1$ supersymmetric gauge theories. In general, the gauge theory curve is realized as a k-sheeted covering of the plane. One of these sheets is visible in the classical theory, while the remaining sheets are not accessible semiclassically bu only in the full quantum theory. A convenient method to compute this curve is by the DV prescription, that relies on the planar limit of a related matrix model or, correspondingly, on solving a set of anomaly equations in the gauge theory. We considered theories with matter content such that, once we fix the number of unbroken U(1) gauge groups, there is no order parameter to distinguish the various classical vacua, hence we denoted the different kinds of classical solutions as branches. Our proposal is that, under these circumstances, there is a one to one correspondence between the number of branches and the degree of the curve.

This proposal holds trivially in the case of ordinary SQCD and has been verified also for SQCD with one adjoint chiral superfield in [5]. In this paper, we have worked out the classical and quantum theory of SQCD with two adjoints and superpotential (2.1) and we have verified that the proposal works also in this case. In particular, we have shown that this theory has three classical vacua, namely the pseudoconfining, the abelian higgs and the nonabelian higgs ones. We have proven that in the quantum theory we can associate

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each sheet of the cubic curve to each of these three branches by looking at the singularities of some meromorphic functions on the curve. Moreover, we have argued that one can interpolate continuously between all the classical vacua with the same number of unbroken U(1) factors. It would be interesting to verify our conjecture for other gauge theories with a higher degree DV curve, in particular one can address the following cases.

Consider a $U(N_c)$ gauge theory with one adjoint and an additional chiral superfield in the symmetric (or antisymmetric) representation. Its DV curve is a cubic, as in our two adjoint SQCD, and has been computed in [15, 16]. One could couple this theory to matter in the fundamental representation and find out the classical branches. According to our proposal, we expect to see, in addition to the pseudoconfining vacua of [15, 16], two different higgs vacua and, in the quantum theory, we expect the three branches (with the same number of unbroken U(1)s) to be connected continuously.

The second theory is a quiver $\mathrm{SU}(N_c) \times \mathrm{SU}(N_c)$ gauge theory with matter in the bifundamental representation. The curve of this theory is again a cubic [16], but it has a weird feature, namely each node of the quiver sees a particular sheet as its own physical sheet and the leftover sheet seems mysterious. It would be interesting to add fundamental matter to this theory and classify its classical branches, then study the quantum theory and see how we can connect the different branches by moving the poles between the sheets. In this way one could clarify our proposal in the case of a quiver theory. Moreover, a Seiberg dual theory to this quiver with fundamentals has been discussed in [17]. It would be nice to see the dual description of the electric branches on the DV curve, which is the same for both dual pairs, along the lines of [10].

The E_n type SQCD

Finally, an extremely interesting theory where to test our proposal is SQCD with two adjoints and E_n type (according to the ADE classification of [11]) tree level superpotential. For instance, one can consider the E_6 theory with superpotential $W = \mathrm{Tr}\, Y^3 + \mathrm{Tr}\, X^4$ deformed by lower dimensional operators. The classical vacua of this theories are not known. However, by studying their flows in connection with the a theorem, [11] argued that there are an infinite number of irreps of the equations of motion with vanishing fundamentals (which we called the pseudoconfining branch). First of all, it would be nice to see explicitly whether the number of pseudoconfining vacua is actually infinite. One could find also the higgs vacua and classify all the branches of the theory, then compute the $\mathcal{N} = 1$ curve and verify if the degree of the curve agrees with the number of branches.

As a byproduct of this analysis, one would shed light on the following mystery. The analytic structure of an $\mathcal{N} = 1$ curve is such that, on the physical sheet, the number of branch cuts are in correspondence with the classical pseudoconfining vacua and, in the classical limit, each branch cut shrinks to a point corresponding to a pseudoconfining vacuum. In this case, if the number of pseudoconfining vacua is infinite, it is not at all clear what the curve would look like, since we would expect an infinite number of branch cuts on the physical sheet. Moreover, one could consider the geometric engineering of this theory as a type IIB superstring theory on a certain local Calabi Yau threefold, in the framework of [7, 8].[17] The classical theory is described by the geometry of a P^1 bundle over a particular ALE space, which is the resolution of an E_n singularity. The classical pseudoconfining vacua of the gauge theory should be seen in the geometry as the compact holomorphic curves of the threefold. According to the geometric transition conjecture, in the quantum theory these holomorphic curves are replaced by three spheres, whose volume is proportional to the gauge theory glueballs. But if we have an infinite number of pseudoconfining vacua, as argued by [11], it is not at all clear how to make sense of the classical geometric picture in the first place, whether there are an infinite number of holomorphic curves in the resolved geometry and, finally, how to perform the blow down map, if any, and compute the deformed Calabi Yau.

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A. Anomaly equations

In this appendix we derive the various anomaly equations we used above to solve for the resolvents. First we get the cubic equation (3.1) satisfied by the gauge theory resolvent $R(x)$, following Ferrari who solved for the planar limit of the related two matrix model [8]. We will show the semiclassical expressions of the resolvent on the different sheets. Then, we will solve for the resolvent $T(x)$ in (5.1). Finally, we write down the recursion relations for the $R(y)$.

A.1 The curve

There are many variations that one can try, but only very few of them are useful. In particular, Ferrari [8] has shown that the following three variations

$$\begin{aligned}
&1st : \delta X = 0\,, && \delta Y = -\frac{1}{32\pi^2}\frac{W_\alpha W^\alpha}{x-X}\,,\\
&2nd : \delta X = -\frac{1}{32\pi^2}\frac{W_\alpha W^\alpha}{x-X}\frac{1}{y-Y}\,, && \delta Y = 0\,,\\
&3rd : \delta X = 0\,, && \delta Y = -\frac{1}{32\pi^2}W_\alpha W^\alpha\frac{1}{x-X}\frac{1}{y-Y}\frac{1}{-x-X}\,. \qquad \text{(A.1)}
\end{aligned}$$

give the following anomalous Ward identity

$$\begin{aligned}
&1st : \lambda R_1(x) = \lambda\frac{\widetilde{S}}{x} - \frac{\alpha}{2x}R(x)\,,\\
&2nd : \left[V'(x) - \hbar R(x) + \lambda y^2\right] Z(x,y) = \lambda y R(x) + \lambda R_1(x) -
\end{aligned}$$

[17] Some problems about the engineering of the Yukawa coupling to the fundamentals have been outlined in [12]. Anyway one can consider the theory without fundamentals and discuss the pseudoconfining vacua to begin with.

$$
- \frac{1}{32\pi^2}\left\langle \mathrm{Tr}\, W_\alpha W^\alpha \frac{V'(x)-V'(X)}{x-X}\frac{1}{y-Y}\right\rangle, \qquad (A.2)
$$

$$
3rd\,:\, Z(x,y)Z(-x,y) = \lambda\left[R(x)+R(-x)\right] - \left(\lambda y + \frac{\alpha}{2x}\right) Z(x,y) - \left(\lambda y - \frac{\alpha}{2x}\right) Z(-x,y)\,,
$$

where $R_k(x)$ are the generalized resolvent in (2.16), $Z(x,y)$ is the chiral operator in (2.14), and $\widetilde{S} = -\frac{1}{32\pi^2}\langle W_\alpha W^\alpha Y\rangle$. By expanding the loop equations (A.3) in powers of y we find the recursion relations

$$
\lambda R_{k+2}(x) = \left[\hbar R(x) - V'(x)\right] R_k(x) + +F_k(x)\,, \qquad (A.3)
$$

$$
\begin{aligned}
&\lambda\left[R_{q+2}(x)+R_{q+2}(-x)\right] + \\
&+\frac{\alpha}{2x}\left[R_{q+1}(x)-R_{q+1}(-x)\right] + \\
&\quad +\hbar \sum_{k+k'=q} R_k(x)R_k(-x) = 0\,,
\end{aligned} \qquad (A.4)
$$

for $k \geq 0$, recalling the definition (3.3) of the quantum deformations $F_k(x)$. The strategy is to plug (A.3) into (A.4) and get at $k=0$ an equation for $R(-x)$, then at $k=2$ use it to obtain the closed equation in $R(x)$. By introducing $\widetilde{w} = \hbar R(x) - V'(x)$ we get the following cubic equation

$$
\widetilde{w}^3 + \widetilde{a}(x^2)\widetilde{w}^2 + \widetilde{b}(x^2)\widetilde{w} + \widetilde{c}(x^2) = 0\,, \qquad (A.5)
$$

where the coefficients are

$$
\begin{cases}
\widetilde{a}(x) = V'(x) + V'(-x) - \frac{\alpha^2}{4\lambda}, \\
\widetilde{b}(x) = V'(x)V'(-x) - \frac{\alpha^2}{4\lambda}[V'(x)+V'(-x)] + \hbar\left[F_0(x) + F_0(-x) + \frac{\alpha\widetilde{S}}{x^2}\right], \\
\widetilde{c}(x) = -\frac{\alpha^2}{4\lambda}V'(x)V'(-x) + \\
\qquad +\hbar\Bigg(F_0(-x)V'(x) + F_0(x)V'(-x) + \lambda\left[F_2(x)+F_2(-x) - \hbar\frac{\widetilde{S}^2}{x^2}\right] + \\
\qquad\quad +\frac{\alpha}{2x}\left[\frac{\widetilde{S}}{x}[V'(x)+V'(-x)] + F_1(x) - F_1(-x)\right]\Bigg).
\end{cases} \qquad (A.6)
$$

Since in the coefficients there appear negative powers of x, we have to rescale the equation (A.5) by multiplying by x^2. Setting $w = x^2\widetilde{w}$ we find our cubic equation (3.1), where the new coefficients are (3.2).

A.2 Semiclassical resolvents

The semiclassical expansion of the resolvent $R(x)$ on the three sheets is

$$
\begin{aligned}
R^I(x) &= \frac{x^2F_0(x)+\alpha\widetilde{S}/2}{p(x)} + \lambda x\frac{2\widetilde{F}_2(x^2)+\alpha\widetilde{F}_1(x^2)}{2v_-(x^2)p(x)} + \mathcal{O}(\hbar)\,, \\
\hbar R^{II}(x) &= V'(x) + \frac{\alpha^2}{4\lambda x^2} - \hbar\frac{x^2F_0(x)+\alpha\widetilde{S}/2}{p(x)} - \hbar\frac{x^2F_0(-x)+\alpha\widetilde{S}/2}{p(-x)} -
\end{aligned}
$$

$$-\hbar\lambda x^2 \frac{2\widetilde{F}_2(x^2) - \alpha\widetilde{F}_1(x^2)}{p(x)p(-x)} + \mathcal{O}(\hbar^2)\,, \tag{A.7}$$

$$\hbar R^{III}(x) = V'(x) - V'(-x) + \hbar\frac{x^2 F_0(-x) + \frac{\alpha\widetilde{S}}{2}}{p(-x)} - \hbar\lambda x \frac{2\widetilde{F}_2(x^2) + \alpha\widetilde{F}_1(x^2)}{2v_-(x^2)p(-x)} + +\mathcal{O}(\hbar^2)\,,$$

where $p(x)$ is the polynomial (2.7), whose roots are the abelian vacua (2.6), and $v_-(x^2)$, whose roots are the nonabelian vacua (2.8), is the odd part of the adjoint polynomial $V'(x)$. By looking at (A.8) we get a quick preview of the structure of the branch points of the gauge theory curve (3.1). Indeed, each pole in the semiclassical resolvent splits up into two branch points in the full quantum theory. If the resolvent has a pole at the same value of x on two different sheets, in the quantum theory a branch cut will appear, connecting those same sheets. Therefore, the branch cuts I_i coming from the splitting of the abelian vacua at the roots of $p(x)$ will connect the first and the second sheet. The branch cuts I_i', symmetric of the former with respect to the origin, i.e. coming from the roots of $p(-x)$, connect the second and the third sheets. The branch cuts $\widehat{I}_i$ and $\widehat{I}_i'$ from the nonabelian vacua, i.e. the roots of $v_-(x^2)$, connect the first and the third sheets. This confirms the analysis of section 3.3.

A.3 The resolvent $T(x)$

Consider the theory (2.1) with meson deformation $m(x) = m_1 + x\,m_2$. Consider the variations (A.1) but drop the field strength factor $-\frac{1}{32\pi^2}W_\alpha W^\alpha$ and find the following three anomaly equations

$$\begin{aligned}
&1st : T_1(x) = -\frac{\alpha}{2\lambda x}T(x)\,,\\
&2nd : \left(V'(x) + \lambda y^2 - \hbar R(x)\right)U(x,y) +\\
&\qquad + m_2 M(x,y) = \hbar T(x)Z(x,y) +\\
&\qquad\qquad + \lambda y T(x) + \lambda T_1(x) + \left\langle \mathrm{Tr}\,\frac{V'(x)-V'(X)}{x-X}\frac{1}{y-Y}\right\rangle,\\
&3rd : \hbar\left[Z(x,y)U(-x,y) +\right.\\
&\qquad \left. + Z(-x,y)U(x,y)\right] = -\lambda y\left[U(x,y) + U(-x,y)\right] -\\
&\qquad\qquad - \frac{\alpha}{2x}\left[U(x,y) - U(-x,y)\right] + \lambda\left[T(x) + T(-x)\right],
\end{aligned} \tag{A.8}$$

where the chiral operators $Z(x,y)$, $U(x,y)$ and $M(x,y)$ are given in (2.14) and we set to zero the terms proportional to u^α, w^α in the supersymmetric vacuum. $T_k(x)$ are the generalized resolvents (2.17). The mesons contribute only through the second anomaly equation with the term proportional to m_2. Recalling the definition (5.3) of the degree $n-1$ polynomials $C_k(x)$, we consider the Laurent expansion in powers of y of the second and third equations in (A.8) and find the recursion relations

$$\lambda T_{k+2}(x) = [\hbar R(x) - V'(x)]T_k(x) - m_2 M_k(x) + \hbar R_k(x)T(x) + C_k(x)\,, \tag{A.9}$$

$$\begin{aligned}
&\lambda\left[T_{k+2}(x) + T_{k+2}(-x)\right] + \frac{\alpha}{2x}\left[T_{k+1}(x) - T_{k+1}(-x)\right] +\\
&\qquad + \hbar\sum_{q+q'=k}\left[R_q(x)T_{q'}(-x) + R_q(-x)T_{q'}(x)\right] = 0\,,
\end{aligned} \tag{A.10}$$

for $k \geq 0$, where $M_k(x)$ are the generalized meson generators in (2.17). Notice that these recursion relations are linear in T_k, whereas the recursion relations for R_k in (A.3) and (A.4) are bilinear. Plugging (A.9) into (A.10) at $k=0$ we solve for $T(-x)$, then at $k=2$ we find a linear equation for $T(x)$, whose solution is precisely

$$T(x) = \frac{N(x) + \delta N(x)}{D(x)} ,$$

where $N, \delta N$ and D are given in (5.2) and (5.4).

A.4 The resolvent $R(y)$

We would like to solve for the resolvent $R(y) = -\frac{1}{32\pi^2} \mathrm{Tr} \frac{W_\alpha W^\alpha}{y-Y}$. There are two methods [8], but we will use the most intuitive one. Consider the anomaly equations (A.3). To solve for $R(x)$ we used their Laurent expansion in y, but now we can use their Laurent expansion in powers of x and find the following recursion relations

$$\textstyle\sum_{i=0}^{n} t_i R_i(y) + \lambda y^2 R(y) - \lambda y S - \lambda \widetilde{S} = 0 \,, \qquad (A.11)$$

$$\textstyle\sum_{i=0}^{n} t_i R_{k+i+1}(y) - \hbar \sum_{i=0}^{k} \widehat{S}_i R_{k-i}(y) + \lambda y^2 R_{k+1}(y) - \lambda y \widehat{S}_{k+1} + \frac{\alpha}{2} \widehat{S}_k = 0 \,, \qquad (A.12)$$

$$2\lambda y R_{2k+1}(y) = 2\lambda \widehat{S}_{2k+1} + \hbar \textstyle\sum_{i=0}^{2k} (-1)^i R_i(y) R_{2k-i}(y) - \alpha R_{2k}(y) \,, \qquad (A.13)$$

for $k \geq 0$, where $R_k(y) = -\frac{1}{32\pi^2} \mathrm{Tr} \frac{W_\alpha W^\alpha}{y-Y} X^k$ are the generalized resolvents and $\widehat{S}_k = -\frac{1}{32\pi^2} \mathrm{Tr}\, W_\alpha W^\alpha X^k$. If we combine these three equations we get a closed degree 2^n algebraic equation for $R(y)$.

B. Solution of the cubic equation

Consider the cubic equation

$$w^3 + a w^2 + b w + c = 0. \qquad (B.1)$$

First get rid of the subleading term by the shift $w = z - a/3$, obtaining $z^3 + 3\gamma z + 2\delta = 0$, where we introduced $3\gamma = b - \frac{a^2}{3}$ and $2\delta = c - \frac{ab}{3} + \frac{2}{27} a^3$. Now the trick is to replace z with two variables under a useful constraint. Set $z = u + v$ and get $u^3 + v^3 + 3(u+v)(uv+\gamma) + 2\delta = 0$. If we just choose $uv + \gamma = 0$ then we get

$$\left\{ u^3 + v^3 + 2\delta = 0 \,, uv + \gamma = 0 \,. \right. \qquad (B.2)$$

We solve the quadratic equation $u^6 + 2\delta u^3 - \gamma^3 = 0$, obtaining $u^3 = -\delta + \sqrt{\delta^2 + \gamma^3}$. The solutions for u picks up the three cubic roots of unity, 1, $e^{i\frac{2\pi}{3}}$ and $e^{-i\frac{2\pi}{3}}$, obtaining

$$\begin{aligned} u^{(I)} &= (-\delta + \sqrt{\delta^2 + \gamma^3})^{\frac{1}{3}} \,, \\ u^{(II)} &= e^{i\frac{2\pi}{3}} u^{(I)} \,, \\ u^{(III)} &= e^{-i\frac{2\pi}{3}} u^{(I)} \,, \end{aligned}$$

and analogous solutions for $v = -\frac{\gamma}{u}$. The solutions to (B.1) are therefore $w^{(i)} = u^{(i)} - \frac{\gamma}{u^{(i)}} - \frac{a}{3}$, that we can list

$$w^{(I)} = (-\delta + \sqrt{\delta^2 + \gamma^3})^{\frac{1}{3}} - \frac{\gamma}{(-\delta + \sqrt{\delta^2 + \gamma^3})^{-\frac{1}{3}}} - \frac{a}{3} \,,$$

$$
\begin{aligned}
w^{(II)} &= e^{i\frac{2}{3}\pi}(-\delta+\sqrt{\delta^2+\gamma^3})^{\frac{1}{3}} - e^{-i\frac{2}{3}\pi}\frac{\gamma}{(-\delta+\sqrt{\delta^2+\gamma^3})^{-\frac{1}{3}}} - \frac{a}{3}\,, \\
w^{(III)} &= e^{-i\frac{2}{3}\pi}(-\delta+\sqrt{\delta^2+\gamma^3})^{\frac{1}{3}} - e^{i\frac{2}{3}\pi}\frac{\gamma}{(-\delta+\sqrt{\delta^2+\gamma^3})^{-\frac{1}{3}}} - \frac{a}{3}\,.
\end{aligned}
\tag{B.3}
$$

C. Holomorphic differentials

In this appendix we compute a basis for the holomorphic differentials on the curve (3.1). We use the method of divisors in the notations of [18]. Let us denote by

$$
[g] = \frac{P_1^{\alpha_1}\dots P_n^{\alpha_n}}{Q_1^{\beta_1}\dots Q_m^{\beta_m}}\,,
$$

the divisor of g, where P_i is a zero of degree α_i and Q_j is a pole of degree β_j. The degree of the divisor is given by $deg[g] = \sum_i \alpha_i - \sum_j \beta_j$. The Riemann-Roch theorem states the degree of a meromorphic function g is $deg[g] = 0$, while for an abelian differential ω the degree is $deg[\omega] = 2g-2$. We need to compute the divisors of dx, x, $\partial_w f = 3(w(x)^2+\gamma(x^2))$ and $w(x)$.

The differential dx vanishes at the branch points and has a double pole at ∞ on each of the three sheets. If we denote by Δ_B the divisor corresponding to the $6(n+1)$ branch points we find

$$
[dx] = \frac{\Delta_B}{P^2_{\infty_I}P^2_{\infty_{II}}P^2_{\infty_{III}}}\,,
$$

so that $deg[dx] = 6n = 2g-2$. The function x has the following divisor

$$
[x] = \frac{\mathcal{O}_I\mathcal{O}_{II}\mathcal{O}_{III}}{P_{\infty_I}P_{\infty_{II}}P_{\infty_{III}}}\,,
$$

where $\mathcal{O}_i$ represents the origin on each sheet.

Now let us consider $\partial_w f$. It vanishes at the branch point locus Δ_B, while $\partial_w f(x,w)$ $\sim_{x\sim\infty} x^{2(n+2)}$ on each sheet. Thus, by Riemann-Roch, we are missing six zeroes. If we study the asymptotics at small x we find that $\partial_w f_{x,w(x)} \sim x^3$ on the 1st and 3rd sheets while $\partial_w f_{x,w(x)} \sim const$ on the 2nd sheet, so that

$$
[\partial_w f] = \frac{\Delta_B\mathcal{O}_I^3\mathcal{O}_{III}^3}{P^{2(n+2)}_{\infty_I}P^{2(n+2)}_{\infty_{II}}P^{2(n+2)}_{\infty_{III}}}\,. \tag{C.1}
$$

Consider now $w(x)$. We need to study its zeroes for small x, in order to cancel the poles coming from (C.1). For small w we can approximate the curve (3.1) by $b(x)w + c(x) = 0$ so that w vanishes at the roots of $c(x)$ which are not roots of $b(x)$. So we expect a double zero at $x = 0$ and a bunch of $2n$ nonvanishing other zeroes, whose corresponding divisor we denote by C_{2n}. The asymptotic expansion of the solutions $w(x)$ is $w(x) \sim_{x\sim 0} x^2$ on the 1st and 3rd sheets and $w(x) \sim_{x\sim 0} const$ on the 2nd sheet so that its divisor is

$$
[w] = \frac{\mathcal{O}_I^2\mathcal{O}_{III}^2 C_{2n}}{P^{n+2}_{\infty_I}P^{n+2}_{\infty_{III}}}\,.
$$

To build the holomorphic differentials we have to take care of the poles coming from (C.1) at the points $\mathcal{O}_{I,III}$, so that

$$\frac{dx}{\partial_w f}\,, \qquad \frac{xdx}{\partial_w f}\,, \tag{C.2}$$

have triple and double poles, respectively, while

$$\frac{x^2 dx}{\partial_w f}\,, \qquad \frac{wdx}{\partial_w f}\,, \tag{C.3}$$

have just single poles at $\mathcal{O}_I, \mathcal{O}_{III}$. Therefore we have to eliminate (C.2) but we can take a linear combination of (C.3) with vanishing residue. Therefore, a basis for the holomorphic differentials is given by

$$\begin{aligned} &\frac{(c_1 x^2 + c_2 w)dx}{w^2+\gamma(x)}\,, \\ &\frac{x^j dx}{w^2+\gamma(x)}\,, \qquad j = 3, \ldots, 2n+2\,, \\ &\frac{x^k w dx}{w^2+\gamma(x)}\,, \qquad k = 1, \ldots, n\,. \end{aligned} \tag{C.4}$$

We have in total $3n+1 = 3(n+1) - 2 = g$ holomorphic differential as expected.

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